Course MFE/3F Practice Exam 4 – Solutions

Solution 1

D Chapter 20, Prepaid Forward Price of $1

We don't need the information provided in statement (i) to answer this question.

The usual formula for a prepaid forward price is:

$$F_{0,T}^{P}(S) = S(0)e^{-\delta T}$$

Let's find the prepaid forward price of $1, in euros. From the perspective of a euro-denominated investor, the initial asset price is:

$$\frac{1}{x(0)} = \frac{1}{1.30}$$

From the perspective of a euro-denominated investor, the dividend yield of $1 is 6%.

The prepaid forward price of one dollar, in euros, is therefore:

$$\frac{1}{x(0)}e^{-0.06\times 4} = \frac{1}{1.30}e^{-0.06\times 4} = 0.60510$$

The prepaid forward price of $1,000, in euros, is:

$$0.60510 \times 1{,}000 = 605.10$$

Solution 2

B Chapter 20, Itô's Lemma

We have:

$$G(t) = 2e^{Z(t)}$$

The partial derivatives are:

$$G_Z = 2e^Z \qquad G_{ZZ} = 2e^Z \qquad G_t = 0$$

This results in:

$$\begin{aligned} dG(t) &= G_Z dZ + \frac{1}{2}G_{ZZ}(dZ)^2 + G_t dt = 2e^{Z(t)}dZ(t) + \frac{1}{2}(2)e^{Z(t)}\left[dZ(t)\right]^2 + 0dt \\ &= 2e^{Z(t)}dZ(t) + e^{Z(t)}\left[dZ(t)\right]^2 \\ &= 2e^{Z(t)}dZ(t) + e^{Z(t)}dt \qquad \text{(because } [dZ(t)]^2 = dt\text{)} \\ &= G(t)dZ(t) + 0.5G(t)dt \\ &= 0.5G(t)dt + G(t)dZ(t) \end{aligned}$$

Dividing both sides by $G(t)$ results in:

$$\frac{dG(t)}{G(t)} = 0.5dt + dZ(t)$$

Solution 3

D Chapters 11 and 13, Greeks in the Cox-Ross-Rubinstein Model

The values of u and d are:

$$u = e^{\sigma\sqrt{h}} = e^{0.30\sqrt{1}} = 1.34986$$
$$d = e^{-\sigma\sqrt{h}} = e^{0.30\sqrt{1}} = 0.74082$$

The risk-neutral probability of an upward movement is:

$$p^* = \frac{e^{(r-\delta)h} - d}{u-d} = \frac{e^{(0.09-0.05)(1)} - 0.74082}{1.34986 - 0.74082} = 0.49257$$

The stock price tree and its corresponding tree of option prices are:

Stock			American Put		
		91.1059			0.0000
	67.4929			0.9275	
50.0000		50.0000	7.3550		2.0000
	37.0409			**14.9591**	
		27.4406			24.5594

If the stock price initially moves down, then the resulting put price is $14.9591. This price is in bold type above to indicate that it is optimal to exercise early at this node:

$$52 - 37.0409 = 14.9591$$

The exercise value of 14.9591 is greater than the value of holding the option, which is:

$$e^{-0.09(1)}\left[(0.49257)2.0000 + (1-0.49257)(24.5594)\right] = 12.2900$$

The current value of the American option is:

$$e^{-0.09(1)}\left[(0.49257)(0.9275) + (1-0.49257)(14.9591)\right] = 7.3550$$

In the CRR model, the stock price after one up movement and one down movement is equal to the initial stock price. Therefore, the formula for theta can be simplified:

$$\theta(S,0) = \frac{V_{ud} - V - (Sud - S)\Delta(S,0) - \frac{(Sud - S)^2}{2}\Gamma(S,0)}{2h}$$

$$= \frac{V_{ud} - V - (50 - 50)\Delta(S,0) - \frac{(50 - 50)^2}{2}\Gamma(S,0)}{2h}$$

$$= \frac{V_{ud} - V}{2h}$$

The value of theta is:

$$\theta(S,0) = \frac{V_{ud} - V}{2h} = \frac{2.0000 - 7.3550}{2} = -2.6775$$

Solution 4

C Chapter 13, Market-Maker's Profit

We do not have enough information to use the standard formula for the market-maker's profit:

$$\text{Market-maker profit} \approx -\frac{\varepsilon^2}{2}\Gamma_t - h\theta_t - rh\left[S_t\Delta_t - V(t)\right]$$

We can use the Black-Scholes Equation to obtain an expression that is equal to zero:

$$h\left[\frac{\sigma^2 S_t^2}{2}\Gamma_t + rS_t\Delta_t + \theta_t - rV(t)\right] = 0$$

Let's add this expression (which is equal to zero) to the Market-maker's profit to obtain a new formula for the market-maker's profit:

$$\text{Market-maker profit} \approx -\frac{\varepsilon^2}{2}\Gamma_t - h\theta_t - rh\left[S_t\Delta_t - V(t)\right] + h\left[\frac{\sigma^2 S_t^2}{2}\Gamma_t + rS_t\Delta_t + \theta_t - rV(t)\right]$$

$$= \frac{\sigma^2 S_t^2}{2}\Gamma_t h - \frac{\varepsilon^2}{2}\Gamma_t$$

The market-maker's profit is:

$$\text{Market-maker profit} \approx \frac{\sigma^2 S_t^2}{2}\Gamma_t h - \frac{\varepsilon^2}{2}\Gamma_t = \frac{0.28^2 50^2}{2}(0.0553)\frac{1}{365} - \frac{2^2}{2}(0.0553)$$

$$= -0.0958$$

Solution 5

D Chapter 10, Replication

The end-of-year payoffs of the call and put options in each scenario are shown in the table below. The rightmost column is the payoff resulting from buying the put option and selling the call option.

Scenario	End of Year Price of Stock A	End of Year Price of Stock B	$P_A(125)$ Payoff	$C_B(50)$ Payoff	$P_A(125) - C_B(50)$ Payoff
1	\$300	\$0	0	0	0
2	\$100	\$0	25	0	25
3	\$50	\$150	75	100	–25

We need to determine the cost of replicating the payoffs in the rightmost column above. We can replicate those payoffs by determining the proper amount of Stock A, Stock B, and the risk-free asset to purchase.

Let's define the following variables:

A = Number of shares of Stock A to purchase

B = Number of shares of Stock B to purchase

C = Amount to lend at the risk-free rate

Since Stock A pays a \$20 dividend at time 0.5, each share of Stock A that is purchased provides its holder with the final price of Stock A at time 1 plus the accumulated value of \$20. Since Stock B pays continuously compounded dividends of 8%, each share of stock B purchased at time 0 grows to $e^{0.08}$ shares of Stock B at time 1.

We have 3 equations and 3 unknown variables:

Scenario 1: $\left[300 + 20e^{0.12(0.5)}\right]A + 0B + Ce^{0.12} = 0$

Scenario 2: $\left[100 + 20e^{0.12(0.5)}\right]A + 0B + Ce^{0.12} = 25$

Scenario 3: $\left[50 + 20e^{0.12(0.5)}\right]A + 150e^{0.08}B + Ce^{0.12} = -25$

Subtracting the first equation from the second equation allows us to solve for A:

$$\left[100 + 20e^{0.12(0.5)}\right]A - \left[300 + 20e^{0.12(0.5)}\right]A = 25$$

$$A = \frac{25}{100 + 20e^{0.12(0.5)} - 300 - 20e^{0.12(0.5)}} = -\frac{1}{8}$$

The equation associated with Scenario 1 can now be used to find C:

$$\left[300+20e^{0.12(0.5)}\right]A+0B+Ce^{0.12}=0$$

$$C=-\left[300+20e^{0.12(0.5)}\right]Ae^{-0.12}=\left[300+20e^{0.12(0.5)}\right]\frac{1}{8}e^{-0.12}=35.6139$$

We use the equation associated with Scenario 3 to find B:

$$\left[50+20e^{0.12(0.5)}\right]A+150e^{0.08}B+Ce^{0.12}=-25$$

$$-\left[50+20e^{0.12(0.5)}\right]\frac{1}{8}+150e^{0.08}B+35.6139e^{0.12}=-25$$

$$B=-0.3462$$

The cost now of replicating the payoffs resulting from buying the put and selling the call is equal to the cost of establishing a position consisting of A shares of Stock A, B shares of Stock B, and C lent at the risk-free rate. Since the price of Stock A is \$100 and the price of Stock B is \$50, we have:

$$100A+50B+C=100\times\frac{-1}{8}+50\times(-0.3462)+35.6139=5.8055$$

Solution 6

B Chapter 24, Black-Derman-Toy Model

In each column of rates, each rate is greater than the rate below it by a factor of:

$$e^{2\sigma_i\sqrt{h}}$$

Therefore, the missing rate in the third column is:

$$0.1977e^{-2\sigma_i\sqrt{1}}=0.1977\times\frac{0.1977}{0.2665}=0.1467$$

The missing rate in the fourth column is:

$$0.4353e^{-2\sigma_i\sqrt{1}}=0.4353\times\frac{0.1530}{0.2168}=0.3072$$

We do not need to calculate the missing rate in the fourth column because the value of a year-3 caplet does not depend on the interest rate in the fourth year, but we included it here for completeness.

The tree of short-term rates is:

			43.53%
		26.65%	
	20.77%		30.72%
15.00%		19.77%	
	17.35%		21.68%
		14.67%	
			15.30%

The caplet pays off only if the interest rate at the end of the second year is greater than 16.00%. The payoff table is:

			N/A
		8.4090	
	0.0000		N/A
0.0000		3.1477	
	0.0000		N/A
		0.0000	
			N/A

The payments have been converted to their equivalents payable at the end of 2 years. The calculations are shown below:

$$\frac{100\times(0.2665-0.1600)}{1.2665}=8.4090$$

$$\frac{100\times(0.1977-0.1600)}{1.1977}=3.1477$$

The present value of these payments is the value of the 3-year caplet:

$$V_0=E^*\left[V_T\times\prod_{i=0}^{T-1}\frac{1}{(1+r_i)}\right]$$

$$=0.5^2\times\frac{8.4090}{(1.15)(1.2077)}+0.5^2\times\frac{3.1477}{(1.15)(1.2077)}+0.5^2\times\frac{3.1477}{(1.15)(1.1735)}=2.66$$

Solution 7

C Chapter 14, Chooser Options and Delta

The price of the chooser option can be expressed in terms of a call option and put option:

$$\text{Price of Chooser Option}=C_{Eur}\left(S_0,K,T\right)+e^{-\delta(T-t_1)}P_{Eur}\left(S_0,Ke^{-(r-\delta)(T-t_1)},t_1\right)$$

$$\text{Price of Chooser Option}=C_{Eur}\left(80,75,5\right)+e^{-0.10(5-2)}P_{Eur}\left(80,75e^{-(0.058-0.10)(5-2)},2\right)$$

$$\text{Price of Chooser Option}=C_{Eur}\left(80,75,5\right)+0.7408P_{Eur}\left(80,85.0712,2\right)$$

From the equation above, we observe that owning the chooser option is equivalent to owning the 5-year call option and 0.7408 of the 2-year put options. To find the delta of the call option, we must calculate d_1 with a 5-year maturity and $K = 75$:

$$d_1 = \frac{\ln(S/K) + (r - \delta + 0.5\sigma^2)T}{\sigma\sqrt{T}} = \frac{\ln(80/75) + (0.058 - 0.10 + 0.5(0.3)^2)5}{0.3\sqrt{5}} = 0.11857$$

$$N(d_1) = N(0.11857) = 0.54719$$

$$\Delta_{Call} = e^{-\delta T}N(d_1) = e^{-(0.10)5} \times 0.54719 = 0.33189$$

To find the delta of the put option, we must calculate d_1 with a 2-year maturity and $K = 85.0712$:

$$d_1 = \frac{\ln(S/K) + (r - \delta + 0.5\sigma^2)T}{\sigma\sqrt{T}} = \frac{\ln(80/85.0712) + (0.058 - 0.10 + 0.5(0.3)^2)2}{0.3\sqrt{2}}$$

$$= -0.13072$$

$$N(-d_1) = N(0.13072) = 0.55200$$

$$\Delta_{Put} = -e^{-\delta T}N(-d_1) = -e^{-(0.10)2} \times 0.55200 = -0.45194$$

Now we can calculate the delta of a portfolio consisting of one of the call options and 0.7408 of the put options. This is the delta of the chooser option:

$$\Delta_{Chooser} = 1 \times \Delta_{Call} + 0.7408\Delta_{Put} = 0.33189 + 0.7408 \times (-0.45194) = -0.00292$$

Since we wrote 100 of the chooser options, the number of shares of stock that we must purchase to delta-hedge is:

$$100 \times (-0.00292) = -0.292$$

Since the result is negative, we must sell 0.292 shares. This rounds to selling zero shares.

Solution 8

A Chapter 12, Elasticity

The values of d_1 and d_2 are:

$$d_1 = \frac{\ln(S/K) + (r - \delta + 0.5\sigma^2)T}{\sigma\sqrt{T}} = \frac{\ln(75/80) + (0.085 - 0.04 + 0.5 \times 0.28^2) \times 0.75}{0.28\sqrt{0.75}}$$

$$= -0.00573$$

$$d_2 = d_1 - \sigma\sqrt{T} = -0.00573 - 0.28\sqrt{0.75} = -0.24822$$

We have:

$$N(-d_1) = N(0.00573) = 0.50229$$

$$N(-d_2) = N(0.24822) = 0.59802$$

The value of the European put option is:

$$\begin{aligned} P_{Eur} &= Ke^{-rT}N(-d_2) - Se^{-\delta T}N(-d_1) \\ &= 80e^{-0.085(0.75)} \times 0.59802 - 75e^{-0.04(0.75)} \times 0.50229 = 8.3285 \end{aligned}$$

The delta of the put option is:

$$\Delta_{Put} = -e^{-\delta T}N(-d_1) = -e^{-0.04(0.75)} \times 0.50229 = -0.48745$$

The elasticity of the put option is:

$$\Omega = \frac{S\Delta}{V} = \frac{75 \times (-0.48745)}{8.3285} = -4.3896$$

Solution 9

A Chapter 20, Itô's Lemma

The expression for $G(t)$ is:

$$G = S^t$$

The partial derivatives are:

$$\begin{aligned} G_S &= tS^{t-1} \\ G_{SS} &= t(t-1)S^{t-2} \\ G_t &= S^t \ln S \end{aligned}$$

From Itô's Lemma and the multiplication rules, we have:

$$\begin{aligned} dG(t) &= G_S dS(t) + \frac{1}{2}G_{SS}[dS(t)]^2 + G_t dt \\ &= tS^{t-1}dS(t) + 0.5t(t-1)S^{t-2}[dS(t)]^2 + S^t \ln S dt \\ &= tS^{t-1}\left[1.5Sdt + SdZ\right] + 0.5t(t-1)S^{t-2}S^2 dt + S^t \ln S dt \\ &= 1.5tS^t dt + tS^t dZ + 0.5t(t-1)S^t dt + S^t \ln S dt \end{aligned}$$

Since $G = S^t$, let's divide both sides by S^t and reorganize the expression:

$$\begin{aligned} \frac{dG(t)}{G(t)} &= 1.5tdt + tdZ + 0.5t(t-1)dt + \ln S dt \\ &= 1.5tdt + 0.5t^2 dt - 0.5tdt + \ln S dt + tdZ \\ &= \left(0.5t^2 + t + \ln S\right)dt + tdZ \end{aligned}$$

We can find an expression for the natural log of S:

$$G = S^t$$
$$\ln G = t \ln S$$
$$\ln S = \frac{\ln G}{t}$$

Substituting this in for the natural log of S, we have:

$$\frac{dG(t)}{G(t)} = \left(0.5t^2 + t + \frac{\ln G(t)}{t}\right)dt + tdZ$$

Solution 10

B Chapter 9, Put-Call Parity

From put-call parity:

$$C_{Eur}(K,T) + Ke^{-rT} = S_0 + P_{Eur}(K,T)$$
$$Ke^{-rT} = S_0 + P_{Eur}(K,T) - C_{Eur}(K,T)$$
$$90e^{-3r} = 75 + 4.11$$
$$e^{-3r} = 0.879$$
$$-3r = \ln(0.879)$$
$$r = 0.04299$$

The continuously compounded risk-free interest rate is 4.30%.

Solution 11

D Chapter 9, Propositions

Dewey is correct, because the prices of the 70-strike and 75-strike puts violate Proposition 2. According to Proposition 2, if the following is violated, then arbitrage is available:

$$P_{Eur}(K_2) - P_{Eur}(K_1) \leq (K_2 - K_1)e^{-rt}$$

Proposition 2 is violated since:

$$P_{Eur}(75) - P_{Eur}(70) > (75.00 - 70.00)e^{-0.07\times 1}$$
$$13.50 - 8.75 > (75.00 - 70.00)e^{-0.07\times 1}$$
$$4.75 > 4.66$$

Since Proposition 2 is violated, arbitrage can be obtained by entering into a put bull spread consisting of buying the 70-strike put and selling the 75-strike put. This results in a positive cash flow at time 0 of:

$$P(75) - P(70) = 4.75$$

The positive cash flow can be lent, resulting in the payoff table below.

		Time 1		
Transaction	**Time 0**	$S_1 < 70$	$70 \le S_1 \le 75$	$75 < S_1$
Buy $P(70)$	–8.75	$70 - S_1$	0	0
Sell $P(75)$	13.50	$-(75 - S_1)$	$-(75 - S_1)$	0
Lend 4.75	–4.75	5.09	5.09	5.09
Total	0.00	0.09	$S_1 - 69.91$	5.09

The bottom row of the payoff table contains some positive values and no negative values, so the strategy produces arbitrage profits.

Louie is also correct, because the call prices violate Proposition 3. According to Proposition 3, if the following is violated, then arbitrage profits are available:

$$\frac{C(K_1) - C(K_2)}{K_2 - K_1} \ge \frac{C(K_2) - C(K_3)}{K_3 - K_2}$$

Proposition 3 is violated, since:

$$\frac{C(60) - C(70)}{70 - 60} < \frac{C(70) - C(75)}{75 - 70}$$
$$\frac{7 - 5}{70 - 60} < \frac{5 - 3}{75 - 70}$$
$$0.2 < 0.4$$

Since Proposition 3 is violated, arbitrage can be obtained by entering into an asymmetric butterfly spread, where λ 60-strike call options are purchased for each 70-strike call option sold and:

$$\lambda = \frac{K_3 - K_2}{K_3 - K_1} = \frac{75 - 70}{75 - 60} = \frac{1}{3}$$

Arbitrage is earned if we create an asymmetric butterfly spread by purchasing $\frac{1}{3}$ of a 60-strike call option, selling 1 70-strike call option, and purchasing $\frac{2}{3}$ of a 75-strike call option. We can scale this up by multiplying by 6, giving us the strategy outlined in the question: purchase 2 of the 60-strike calls, sell 6 of the 70-strike calls, and purchase 4 of the 75-strike calls. This results in a positive cash flow at time 0 of:

$$-2 \times 7.00 + 6 \times 5.00 - 4 \times 3.00 = 4.00$$

The positive cash flow can be lent, resulting in the payoff table below.

		Time 1			
Transaction	**Time 0**	$S_1 < 60$	$60 \le S_1 \le 70$	$70 \le S_1 \le 75$	$75 < S_1$
Buy 2 of $C(60)$	$-2(7.00)$	0.00	$2(S_1 - 60)$	$2(S_1 - 60)$	$2(S_1 - 60)$
Sell 6 of $C(70)$	$6(5.00)$	0.00	0.00	$-6(S_1 - 70)$	$-6(S_1 - 70)$
Buy 4 of $C(75)$	$-4(3.00)$	0.00	0.00	0.00	$4(S_1 - 75)$
Lend 4.00	-4.00	4.29	4.29	4.29	4.29
Total	0.00	4.29	$2S_1 - 115.71$	$304.29 - 4S_1$	4.29

The bottom row of the payoff table contains some positive values and no negative values, so the strategy produces arbitrage profits.

Solution 12

D Chapter 9, Factors Affecting Premiums

Statement I is false, because if the time to maturity is increased so that it occurs after a liquidating dividend are paid, then the European call option expires worthless. See page 297 of the textbook.

Statement II is true, because before expiration an American call option on a non-dividend paying stock can be sold for more than its exercise value. See page 294 of the textbook.

Statement III is true because an American option can always be turned into a shorter-lived American option. See page 293 of the textbook.

Solution 13

A Chapter 21, Black-Scholes Equation

We can use the Black-Scholes equation to find rV:

$$0.5\sigma^2 S^2 V_{SS} + (r - \delta)SV_S + V_t = rV$$

$$0.5(0.30)^2(5)^2(0.245) + (0.08 - 0.02)(5)(0.624) - 0.405 = rV$$

$$0.057825 = rV$$

The expected change per unit of time under the risk-neutral distribution is:

$$\frac{E^*[dV]}{dt} = rV = 0.057825$$

Solution 14

D Chapter 20, Multiplication Rules

To keep the presentation from becoming too cluttered, we drop the functional relationships and conditional statement from the notation until the end.

We have:

$$dX(t) = 0.05Xdt + 0.40XdZ$$
$$dQ(t) = 0.13Qdt + 0.20QdZ'$$

We begin by multiplying the expression out:

$$E\left[dX(t)\times dQ(t)\middle|X(t),Q(t)\right]$$
$$= E\left[(0.05Xdt + 0.40XdZ)\times(0.13Qdt + 0.20QdZ')\right]$$
$$= E\left[0.05Xdt(0.13)Qdt + 0.40X(0.13)QdtdZ + 0.05Xdt(0.20)QdZ' + 0.40X(0.20)QdZdZ'\right]$$

The first 3 terms in the expression become zero because of the following multiplication rules:

$$(dt)^2 = 0$$
$$dt\times dZ = 0$$
$$dt\times dZ' = 0$$

For the fourth term, we make use of the following multiplication rule:

$$dZ\times dZ' = \rho dt$$

After applying the multiplication rules, we have:

$$E\left[0+0+0+0.40(0.20)dZdZ'XQ\right]$$
$$= E\left[0+0+0+0.40(0.20)(0.8dt)XQ\right]$$
$$= 0.064XQdt$$

Including the functional relationships, this is written as:

$$0.064X(t)Q(t)dt$$

Solution 15

E Chapter 12, Black-Scholes Formula for Currencies

Brad wants to give up \$62,500 and receive €50,000 at the end of 6 months. This can be accomplished with either of 2 different purchases:

- 50,000 dollar-denominated call options on euros, each with a strike price of \$1.25.
- 62,500 euro-denominated put options on dollars, each with a strike price of €0.80.

Therefore, the correct answer must be either Choice B or Choice E.

Let's find the price of the dollar-denominated call options.

The values of d_1 and d_2 are:

$$d_1 = \frac{\ln(x_0 / K) + (r - r_f + 0.5\sigma^2)T}{\sigma\sqrt{T}} = \frac{\ln(1.25/1.25) + (0.07 - 0.04 + 0.5 \times 0.12^2)(0.5)}{0.12\sqrt{0.5}}$$
$$= 0.21920$$
$$d_2 = d_1 - \sigma\sqrt{T} = 0.21920 - 0.12\sqrt{0.5} = 0.13435$$

We have:

$$N(d_1) = N(0.21920) = 0.58675$$

$$N(d_2) = N(0.13435) = 0.55344$$

The value of the call option is:

$$C_{Eur} = x_0 e^{-r_f T} N(d_1) - K e^{-rT} N(d_2)$$
$$= 1.25e^{-0.04(0.5)} \times 0.58675 - 1.25e^{-0.07(0.5)} \times 0.55344 = 0.050909$$

The value of 50,000 of the dollar-denominated call options is:

$$\$0.050909 \times 50,000 = \$2,545.43$$

Therefore, Choice B is not correct.

The 50,000 dollar-denominated calls with a strike price of \$1.25 (described in Choice B) allow their owner the right to give up \$62,500 to obtain €50,000. Likewise, 62,500 euro-denominated puts with a strike price of €0.80 (described in Choice E) allow their owner the right to give up \$62,500 to obtain €50,000. Since both option positions have the same payoff, they must have the same current cost:

$$\frac{\$2,545.43}{\$1.25/€} = €2,036.35$$

Therefore, Choice E is correct.

Alternatively, we can use a relationship from the Chapter 9 Review Note to find the cost of one of the euro-denominated put options:

$$C_{\$}(x_t, K, T-t) = x_t K P_{€}\left(\frac{1}{x_t}, \frac{1}{K}, T-t\right)$$
$$0.050909 = 1.25 \times 1.25 \times P_{€}(0.80, 0.80, 0.5)$$
$$P_{€}(0.80, 0.80, 0.5) = 0.032582$$

The value of 62,500 of the put options is therefore:

$$62,500 \times €0.032582 = €2,036.35$$

Solution 16

A Chapter 12, Options on Currencies

Since the call option is euro-denominated, we use the euro as the base currency. This means that the current exchange rate is:

$$x_0 = \frac{1}{135}$$

We also have:

$$r = 0.05$$
$$r_f = 0.015$$

The values of d_1 and d_2 are:

$$d_1 = \frac{\ln(x_0 / K) + (r - r_f + 0.5\sigma^2)T}{\sigma\sqrt{T}} = \frac{\ln\left(\frac{1/135}{0.007}\right) + (0.05 - 0.015 + 0.5 \times 0.12^2) \times 0.25}{0.12\sqrt{0.25}}$$
$$= 1.11867$$
$$d_2 = d_1 - \sigma\sqrt{T} = 1.11867 - 0.12\sqrt{0.25} = 1.05867$$

We have:

$$N(d_1) = N(1.11867) = 0.86836$$

$$N(d_2) = N(1.05867) = 0.85512$$

The value of the call option is:

$$C_{Eur} = x_0 e^{-r_f T} N(d_1) - Ke^{-rT} N(d_2)$$
$$= \frac{1}{135} e^{-0.015(0.25)} \times 0.86836 - 0.007 e^{-0.05(0.25)} \times 0.85512 = 0.0004967$$

Since the call option is denominated in euros, the price of the call option is 0.0004967 euros.

Solution 17

A Chapter 22, Supershares

We do not need to know the dividend yield on the index to solve this problem.

We can establish a system of 3 equations and 3 unknowns that allows us to solve for the supershare prices:

$$e^{-0.15 \times 0.5} = SS_1 + SS_2 + SS_3$$
$$0.25 = SS_1$$
$$104.61 = 135 SS_2 + 175 SS_3$$

The solution to this system is:

$$SS_1 = 0.25$$
$$SS_2 = 0.349877$$
$$SS_3 = 0.327865$$

The solution to the question is:

$$2SS_1 - SS_2 + SS_3 = 2 \times 0.25 - 0.349877 + 0.327865 = 0.478$$

Solution 18

D Chapter 11, Utility Values and State Prices

Since the probability of the high state is 0.60, the probability of the low state is:

$$1 - p = 1 - 0.60 = 0.40$$

The stock's cash flow in the low state can now be determined:

$$S = Q_u S_u e^{\delta h} + Q_d S_d e^{\delta h}$$
$$S = pU_u S_u e^{\delta h} + (1-p)U_d S_d e^{\delta h}$$
$$19.44 = 0.60 \times 0.58 \times 30e^{0\times 3} + 0.40 \times 1.25 \times C_L e^{0\times 3}$$
$$C_L = 18$$

The risk-free rate can be found with the probabilities and utility values:

$$e^{-rh} = Q_u + Q_d$$
$$e^{-rh} = pU_u + pU_d$$
$$e^{-r\times 3} = 0.60 \times 0.58 + 0.40 * 1.25$$
$$r = 0.05496$$

The payoffs of the derivative at time 5 are:

$$V_u(5) = \ln(30^5) = 17.0060$$
$$V_d(5) = \ln(18^5) = 14.4519$$

Although the derivative does not pay until time 5, its payoffs are known with certainty at time 3. Therefore, we can discount the payoffs back to time 3 at the risk-free interest rate.

$$V_u(3) = V_u(5)e^{-r\times 2} = 17.0060e^{-0.05496\times 2} = 15.2358$$
$$V_d(3) = V_d(5)e^{-r\times 2} = 14.4519e^{-0.05496\times 2} = 12.9476$$

The value of the derivative is:

$$V(0) = Q_u V_u(3) + Q_d V_d(3) = pU_u \times V_u(3) + (1-p)U_d \times V_d(3)$$
$$= 0.60 \times 0.58 \times 15.2358 + 0.40 \times 1.25 \times 12.9476 = 11.78$$

Solution 19

E Chapter 24, Black-Derman-Toy Model

The yield volatility for the 3-year bond is:

$$\text{Yield volatility} = 0.5\times\ln\left[\frac{P(1,T,r_u)^{-1/(T-1)}-1}{P(1,T,r_d)^{-1/(T-1)}-1}\right] = 0.5\times\ln\left[\frac{P(1,3,r_u)^{-1/2}-1}{P(1,3,r_d)^{-1/2}-1}\right]$$

$$= 0.5\times\ln\left[\frac{0.7560^{-1/2}-1}{0.8099^{-1/2}-1}\right] = 0.1501$$

Solution 20

D Chapter 22, All-or-Nothing Options

The trick to answering this question quickly is recognizing that d_1 for Stock A is equal to d_2 for Stock B:

$$\text{Stock A: } d_1 = \frac{\ln\left(S_t/K\right)+[r-\delta+0.5\sigma^2](T-t)}{\sigma\sqrt{T-t}}$$

$$= \frac{\ln\left(75/K\right)+[0.10-0.09+0.5(0.2)^2](2)}{0.2\sqrt{2}}$$

$$= \frac{\ln\left(75/K\right)+[0.03](2)}{0.2\sqrt{2}}$$

$$\text{Stock B: } d_2 = \frac{\ln\left(S_t/K\right)+[r-\delta-0.5\sigma^2](T-t)}{\sigma\sqrt{T-t}}$$

$$= \frac{\ln\left(75/K\right)+[0.10-0.05-0.5(0.2)^2](2)}{0.2\sqrt{2}}$$

$$= \frac{\ln\left(75/K\right)+[0.03](2)}{0.2\sqrt{2}}$$

For Stock B, we can use the cash call price to determine $N(d_2)$:

$$CashCall(K) = e^{-r(T-t)}N(d_2)$$

$$0.39 = e^{-0.10(2)}N(d_2)$$

$$N(d_2) = 0.47635$$

This value of $N(d_2)$ replaces $N(d_1)$ in the formula for the asset put on Stock A:

$$\begin{aligned} AssetPut(K) &= S_t e^{-\delta(T-t)} N(-d_1) \\ &= 75e^{-0.09(2)}[1 - N(d_1)] \\ &= 75e^{-0.09(2)}[1 - 0.47635] \\ &= 32.80 \end{aligned}$$

Solution 21

B Chapter 19, Control Variate Estimate

The payoff of an average strike Asian call option is the final stock price minus the average stock price, when that amount is greater than zero:

Average strike Asian call option payoff = $\text{Max}[S_T - \bar{S}, 0]$

These payoffs are shown in the rightmost columns below:

i	Arithmetic Average	Geometric Average	Final Stock Price	$Y_i e^{0.10}$ Arithmetic Option Payoff	$X_i e^{0.10}$ Geometric Option Payoff
1	45.70	45.40	43.70	0.00	0.00
2	99.60	96.40	133.80	34.20	37.40
3	38.00	37.80	43.80	5.80	6.00
4	37.50	36.90	28.40	0.00	0.00

The values in the two rightmost columns are the payoffs, which means that they are the discounted Monte Carlo prices, Y_i and X_i, times e^{rT}.

The estimate for β is:

$$\beta = \frac{\sum_{i=1}^{n}(Y_i - \bar{Y})(X_i - \bar{X})}{\sum_{i=1}^{n}(X_i - \bar{X})^2} = \frac{\left(e^{rT}\right)^2 \sum_{i=1}^{n}(Y_i - \bar{Y})(X_i - \bar{X})}{\left(e^{rT}\right)^2 \sum_{i=1}^{n}(X_i - \bar{X})^2}$$

We get the same estimate for β regardless of whether we use the time 0 prices or the time 1 payoffs (as shown in the rightmost expression above). To save time, we use the time 1 payoffs from the table above.

We perform a regression using the sixth column in the table above as the x-values and the fifth column as the y-values. The resulting slope coefficient is:

$\beta = 0.912861693$

During the exam, it is more efficient to let the calculator perform the regression and determine the slope coefficient.

Using the TI-30XS Multiview calculator, we first clear the data by pressing [data] [data] ↓↓↓ until Clear ALL is shown, and then press [enter].

Fill out the table as shown below:

L1	L2	L3
0.00	0.00	-----------
37.40	34.20	
6.00	5.80	
0.00	0.00	

Next, press:

[2nd] [quit] (to exit the table)

[2nd] [stat] 2 (i.e., select 2-Var Stats)

Select L1 for x-data and press [enter].

Select L2 for y-data and press [enter].

Select CALC and then [enter]

Exit the data table by pressing [2nd] [quit].

Obtain access to the statistics by pressing [2nd] [stat] 3

Note the following statistics:

$\bar{x} = 10.85$

$\bar{y} = 10$

$a = 0.912861693$ (This is β)

Alternatively, the TI-30X IIS or the BA II Plus can be used to obtain the values above. To use one of those calculators, follow the steps outlined at the end of this solution and then return to this point to continue the solution.

The Monte Carlo estimate for the arithmetic average strike Asian put is:

$$\bar{A} = \bar{Y} = \frac{e^{-r(T-t)}}{n}\sum_{i=1}^{n} V_i(T) = e^{-0.10\times(1-0)}\left[10\right] = 9.0484$$

The Monte Carlo estimate for the geometric average strike Asian put is:

$$\bar{G} = \bar{X} = \frac{e^{-r(T-t)}}{n}\sum_{i=1}^{n} V_i(T) = e^{-0.10\times(1-0)}\left[10.85\right] = 9.8175$$

The true price for the geometric option is \$4.50, so the control variate price of the arithmetic average option is:

$$A^* = \bar{A} + \beta(G - \bar{G}) = 9.0484 + 0.91286(4.50 - 9.8175) = 4.1942$$

<u>Alternative Calculators</u>

Using the TI-30X IIS, the procedure is:

[2nd][STAT] (Select 2-VAR) [ENTER]

[DATA]

X1= 0.00 ↓ (Hit the down arrow once)

Y1= 0.00 ↓

X2= 37.40 ↓

Y2= 34.20 ↓

X3= 6.00 ↓

Y3= 5.80 ↓

X4= 0.00 ↓

Y4= 0.00 [ENTER]

[STATVAR]

Press the right arrow and note the following statistics:

$\bar{x} = 10.85$

$\bar{y} = 10$

$a = 0.912861693$ (This is β)

To exit the statistics mode: [2nd] [EXITSTAT] [ENTER]

Using the BA II Plus calculator, the procedure is:

[2nd][DATA] [2nd][CLR WORK]

X01= 0.00 [ENTER] ↓ (Hit the down arrow once)

Y01= 0.00 [ENTER] ↓

X02= 37.40 [ENTER] ↓

Y02= 34.20 [ENTER] ↓

X03= 6.00 [ENTER] ↓

Y03= 5.80 [ENTER] ↓

X04= 0.00 [ENTER] ↓

Y04= 0.00 [ENTER]

[2nd][STAT]

Press the down arrow and note the following statistics:

$\bar{x} = 10.85$

$\bar{y} = 10$

$b = 0.91286169$ (This is β)

To exit the statistics mode: [2nd][QUIT]

Solution 22

B Chapters 10 & 24, Risk-Neutral Probability

Let's use $P(0,2)$ to denote the price of a 2-year zero-coupon bond that matures for $1.

We can make use of put-call parity:

$$C(90) + 90 \times P(0,2) = S_0 + P(90)$$
$$C(90) + 90 \times P(0,2) = 80 + P(90)$$
$$C(90) - P(90) = 80 - 90 \times P(0,2)$$

We can use the stock prices to determine the risk-neutral probability that the up state of the world occurs:

$$p^* = \frac{e^{(r-\delta)h} - d}{u-d} = \frac{(1+r_0)^h - d}{u-d} = \frac{(1.07)^1 - 40/80}{120/80 - 40/80} = 0.57$$

If the up state occurs, then the zero-coupon bond will have a value of $\frac{1}{1.10}$ at time 1, and if the down state occurs, then the zero-coupon bond will have a value of $\frac{1}{1.051}$ at time 1. The time 0 value is found using the risk-neutral probabilities and the risk-free rate at time 0:

$$P(0,2) = \frac{1}{1.07}\left[0.57 \times \frac{1}{1.10} + 0.43 \times \frac{1}{1.051}\right] = 0.86665$$

We can now use the equation for put-call parity described above to find the solution:

$$C(90) - P(90) = 80 - 90 \times P(0,2) = 80 - 90 \times 0.86665 = 2.00$$

Solution 23

D Chapter 18, Partial Expectation

The values of $\hat{d}_1$ (rounded to two places) and $N(-\hat{d}_1)$ are:

$$\hat{d}_1 = \frac{\ln\left(\frac{S_0}{S_0}\right) + (\alpha - \delta + 0.5\sigma^2)(T-t)}{\sigma\sqrt{T-t}} = \frac{0 + (0.10 - 0.04 + 0.5(0.25)^2)(3)}{0.25\sqrt{3}} = 0.63220$$

$$N(-\hat{d}_1) = N(-0.63220) = 0.26363$$

We can use the formula for the partial expectation to solve for the current stock price:

$$PE\left[S_T \middle| S_T < K\right] = S_t e^{(\alpha-\delta)(T-t)} N(-\hat{d}_1)$$

$$PE\left[S_3 \middle| S_3 < S_0\right] = S_0 e^{(0.10-0.04)3} N(-\hat{d}_1)$$

$$27.67 = S_0 e^{(0.10-0.04)3} \times 0.26363$$

$$S_0 = 87.6680$$

Solution 24

C Chapter 14, Rolling Insurance Strategy

The values of d_1 and d_2 for a 3-month put option with a strike price that is 95% of the current stock price are:

$$d_1 = \frac{\ln(S/0.95S) + (r - \delta + 0.5\sigma^2)T}{\sigma\sqrt{T}} = \frac{-\ln(0.95) + (0.15 - 0.09 + 0.5 \times 0.4^2)(0.25)}{0.4\sqrt{0.25}}$$

$$= 0.43147$$

$$d_2 = d_1 - \sigma\sqrt{T} = 0.43147 - 0.4\sqrt{0.25} = 0.23147$$

From the normal distribution table:

$$N(-d_1) = N(-0.43147) = 0.33306$$

$$N(-d_2) = N(-0.23147) = 0.40847$$

The cost of a 3-month put option that has a strike price of 95% of the current stock price is a function of the then-current stock price:

$$\begin{aligned} P_{Eur}(S, 0.95S, 0.25) &= 0.95Se^{-rT} N(-d_2) - Se^{-\delta T} N(-d_1) \\ &= 0.95Se^{-0.25\times0.15} N(-d_2) - Se^{-0.25\times0.09} N(-d_1) \\ &= S\left[0.95e^{-0.0375}(0.40847) - e^{-0.0225}(0.33306)\right] \\ &= 0.04811S \end{aligned}$$

Therefore, the first put option can be purchased with 0.04811 shares of stock now, and the value of this purchase price is:

$$0.04811S = 0.04811F_{0,0}^{P}(S)$$

The second option can be purchased with 0.04811 shares of stock in 3 months. The current value of a share of stock 3 months from now is its prepaid forward price, so the time 0 value of this purchase price is:

$$0.04811F_{0,0.25}^{P}(S)$$

The third option can be purchased with 0.04811 shares of stock in 6 months. The current value of a share of stock 6 months from now is its prepaid forward price, so the time 0 value of this purchase price is:

$$0.04811F_{0,0.5}^{P}(S)$$

The fourth option can be purchased with 0.04811 shares of stock in 9 months. The current value of a share of stock 9 months from now is its prepaid forward price, so the time 0 value of this purchase price is:

$$0.04811F_{0,0.75}^{P}(S)$$

Summing these prices, we have:

$$\begin{aligned}
&0.04811\left[F_{0,0}^{P}(S) + F_{0,0.25}^{P}(S) + F_{0,0.5}^{P}(S) + F_{0,0.75}^{P}(S)\right] \\
&= 0.04811\left[100 + 100e^{-0.25\times0.09} + 100e^{-0.5\times0.09} + 100e^{-0.75\times0.09}\right] \\
&= 4.811\left[1 + e^{-0.25\times0.09} + e^{-0.5\times0.09} + e^{-0.75\times0.09}\right] \\
&= 4.811\ddot{a}_{\overline{1}|0.09}^{(4)} = 4.811 \times \frac{1 - e^{-0.09}}{e^{0.25\times0.09} - 1}e^{0.25\times0.09} = 4.811 \times 3.8685 \\
&= 18.6129
\end{aligned}$$

Solution 25

C Chapter 10, Options on Futures Contracts

Since the risk-neutral probability of an up move is equal to the risk-neutral probability of a down move, both probabilities are 50%:

$$p^* = 1 - p^* \quad \Rightarrow \quad p^* = 0.5$$

We are given that the ratio of the factors applicable to the futures price is:

$$\frac{u_F}{d_F} = \frac{3}{2}$$

The formula for the risk-neutral probability of an up move can be used to find u_F and d_F :

$$p^* = \frac{1 - d_F}{u_F - d_F}$$

$$p^* = \frac{\frac{1}{d_F} - \frac{d_F}{d_F}}{\frac{u_F}{d_F} - \frac{d_F}{d_F}}$$

$$\frac{1}{2} = \frac{\frac{1}{d_F} - 1}{\frac{3}{2} - 1}$$

$$d_F = 0.8$$

$$u_F = \frac{3}{2} \times d_F = 1.2$$

The tree of futures prices is therefore:

Futures Prices			
			172.8000
		144.0000	
	120.0000		115.2000
		96.0000	
			76.8000

The tree of prices for the European put option is:

European Put			
			0.0000
		7.0391	
	18.7301		14.8000
		32.3418	
			53.2000

The price of the European put is:

$$e^{-0.10 \times 0.5} \left[(1/2)7.0391 + (1/2)32.3418 \right] = 18.7301$$

The tree of prices for the American put option is:

American Put			
			0.0000
		7.0391	
	19.5188		14.8000
		34.0000	
			53.2000

If the futures price moves down to \$96, then early exercise is optimal, as indicated above by the bolding for that node.

The price of the American put is:

$$e^{-0.10\times0.5}\left[(1/2)7.0391+(1/2)34.0000\right]=19.5188$$

The price of the American put option exceeds the price of the European put option by:

$$19.5188-18.7301=0.7887$$

Solution 26

D Chapter 20, Power Option

Since $S(t)$ is a geometric Brownian motion, so is $[S(t)]^a$:

$$\frac{dS(t)}{S(t)}=(\alpha-\delta)dt+\sigma dZ(t) \quad\Rightarrow\quad \frac{d(S^a)}{S^a}=\left[a(\alpha-\delta)+0.5a(a-1)\sigma^2\right]dt+a\sigma dZ(t)$$

The coefficient of $dZ(t)$ is the asset's volatility. For $[S(t)]^3$, let's call this volatility σ^*:

$$\sigma^*=a\sigma=3\times0.30=0.90$$

Using the expected rate of price appreciation of 11% and the expected return of 15%, we can determine the dividend yield of the stock:

$$0.11=\alpha-\delta$$
$$0.11=0.15-\delta$$
$$\delta=0.04$$

The dividend yield on $[S(t)]^3$ is:

$$\delta^*=r-a(r-\delta)-0.5a(a-1)\sigma^2=0.10-3(0.10-0.04)-0.5(3)(3-1)(0.30)^2=-0.35$$

The initial price of $[S(t)]^3$ is:

$$[S(0)]^3=2^3=8$$

We can use the Black-Scholes formula to find the delta of a 1-year European call option with $[S(t)]^3$ as the underlying asset. Since the underlying asset is $[S(t)]^3$, we must determine the strike price that is compared to $[S(t)]^3$:

$$[S(1)]>2.1 \quad\Rightarrow\quad [S(1)]^3>2.1^3$$
$$K^*=K^3=2.1^3=9.261$$

We can find d_1:

$$d_1=\frac{\ln(S^3/K^*)+\left(r-(\delta^*)+0.5(\sigma^*)^2\right)T}{(\sigma^*)\sqrt{T}}$$
$$=\frac{\ln(8/9.261)+(0.10-(-0.35)+0.5\times0.90^2)\times1}{0.90\sqrt{1}}=0.78737$$

We have:

$$N(d_1) = N(0.78737) = 0.78447$$

The partial derivative of the power option with respect to the underlying asset of $[S(t)]^3$ is:

$$\frac{\partial V}{\partial [S(t)]^3} = e^{-(\delta^*)} N(d_1) = e^{-(-0.35)} \times 0.78447 = 1.1132$$

But we need the partial derivative with respect to the stock price, which is the delta of the power option:

$$\Delta = \frac{\partial V}{\partial S} = \frac{\partial V}{\partial [S(t)]^3} \times \frac{\partial [S(t)]^3}{\partial S} = 1.1132 \times 3[S(t)]^2 = 1.1132 \times 3 \times 2^2 = 13.3586$$

Therefore, the market-maker must purchase 13.3710 shares of stock.

Solution 27

C Chapter 24, Risk-Neutral Version of the Vasicek Model

The process for the short rate follows the Vasicek Model. The true process can be rewritten in the familiar form:

$$dr = 0.06(0.12 - r)dt + 0.07dZ \qquad \Rightarrow \qquad a = 0.06,\ b = 0.12,\ \sigma = 0.07$$

We can use (iii) to obtain the Sharpe ratio:

$$\tilde{Z}(t) = Z(t) - \phi t$$
$$\tilde{Z}(5) = Z(5) - 5\phi$$
$$-0.06 = 0.24 - 5\phi$$
$$\phi = 0.06$$

The parameter $B(5,10)$ is:

$$B(t,T) = \frac{1 - e^{-a(T-t)}}{a}$$
$$B(5,10) = \frac{1 - e^{-a(10-5)}}{a} = \frac{1 - e^{-0.06\times 5}}{0.06} = 4.3197$$

We can use the Sharpe ratio to determine $\alpha(0.09,5,10)$:

$$\phi(r,t) = \frac{\alpha(r,t,T) - r}{q(r,t,T)}$$
$$\phi = \frac{\alpha(r,t,T) - r}{B(t,T)\sigma}$$
$$\alpha(r,t,T) = r + B(t,T)\sigma\phi$$
$$\alpha(0.09,5,10) = 0.09 + 4.3197 \times 0.07 \times 0.06 = 0.1081$$

Solution 28

E Chapter 11, Alternative Binomial Trees

If the option has a positive payoff in both the up and down states, then:

$$\Delta = e^{-\delta h}\frac{V_u - V_d}{S(u-d)} = e^{-0.12\times 0.5}\frac{(53-Su)-(53-Sd)}{Su-Sd} = e^{-0.06}\frac{Sd-Su}{Su-Sd} = -0.9418$$

But $\Delta = -0.64$, so it must be the case that $V_u = 0$.

We can obtain B, the amount of cash that is lent in the replicating portfolio:

$$V = S\Delta + B$$
$$7.75 = 50(-0.64) + B$$
$$B = 39.75$$

We can use the values of Δ and B to solve for u and d:

$$-0.64 = \Delta = e^{-\delta h}\frac{V_u - V_d}{S(u-d)} = e^{-0.12\times 0.5}\frac{0-(53-50d)}{50u-50d} = e^{-0.06}\frac{50d-53}{50(u-d)}$$
$$39.75 = B = e^{-rh}\frac{uV_d - dV_u}{u-d} = e^{-0.07\times 0.5}\frac{u(53-50d)-d\times 0}{u-d} = e^{-0.035}\frac{u(53-50d)}{u-d}$$

Dividing the first equation by the second allows us to solve for u:

$$\frac{-0.64}{39.75} = \frac{e^{-0.06}\dfrac{50d-53}{50(u-d)}}{e^{-0.035}\dfrac{u(53-50d)}{u-d}}$$
$$-0.01610 = \frac{-e^{-0.025}\times\dfrac{1}{50}\times\dfrac{53-50d}{(u-d)}}{u\dfrac{53-50d}{u-d}}$$
$$-0.01610 = -0.01951\times\frac{1}{u}$$
$$u = 1.2115$$

Now we can find the value of d:

$$-0.64 = e^{-0.12\times 0.5}\frac{50d-53}{50(u-d)}$$
$$-0.64 = e^{-0.06}\frac{50d-53}{50(1.2115-d)}$$
$$33.9788d - 41.1659 = 50d - 53$$
$$d = 0.7387$$

The standard binomial model, the Cox-Ross-Rubinestein model and the Jarrow-Rudd model all have the same ratio of u to d:

$$\frac{u}{d} = e^{2\sigma\sqrt{h}}$$
$$\frac{1.2115}{0.7387} = e^{2\sigma\sqrt{0.5}}$$
$$\sigma = 0.3499$$

Solution 29

A Chapter 18, Covariance of S_t and S_T

To answer this question, we use the following formulas:

$$E\left[\frac{S_T}{S_t}\right] = e^{(\alpha-\delta)(T-t)}$$

$$Var\left[S_T \mid S_t\right] = S_t^2 e^{2(\alpha-\delta)(T-t)}\left(e^{\sigma^2(T-t)}-1\right)$$

$$Cov[S_t, S_T] = E\left[\frac{S_T}{S_t}\right] Var\left[S_t \mid S_0\right]$$

The variance of X is:

$$\begin{aligned} Var\left[X \mid S(0)\right] &= Var\left[4\left(2S(1)-S(3)\right)\right] = Var\left[8S(1)-4S(3)\right] \\ &= 64Var[S(1)] + 16Var[S(3)] + 2\times 8\times(-4)Cov[S(1),S(3)] \end{aligned}$$

The variances of $S(1)$ and $S(3)$ are:

$$Var\left[S_1 \mid S_0\right] = 4^2 e^{2(0.07-0.02)(1-0)}\left(e^{0.22^2(1-0)}-1\right) = 0.8769$$

$$Var\left[S_3 \mid S_0\right] = 4^2 e^{2(0.07-0.02)(3-0)}\left(e^{0.22^2(3-0)}-1\right) = 3.3751$$

The covariance is:

$$Cov[S_1, S_3] = E\left[\frac{S_3}{S_1}\right] Var\left[S_1 \mid S_0\right] = e^{(0.07-0.02)(3-1)} \times 0.8769 = 0.9691$$

We can now find the variance of X:

$$\begin{aligned} Var\left[X \mid S(0)\right] &= 64Var[S(1)] + 16Var[S(3)] + 2\times 8\times(-4)Cov[S(1),S(3)] \\ &= 64\times 0.8769 + 16\times 3.3751 + 2\times 8\times(-4)\times 0.9691 \\ &= 48.0992 \end{aligned}$$

Solution 30

D Chapter 19, Stratified Sampling Method

The stratified sampling method assigns the first, fifth, and ninth uniform (0, 1) random numbers to the segment (0.00, 0.25), the second, sixth and tenth uniform (0, 1) random variables to the segment (0.25, 0.50), and the third, seventh and eleventh uniform (0, 1) random variables to the segment (0.50, 0.75), the fourth, eighth and the twelfth uniform (0, 1) random variables to the segment (0.75, 1.00).

The lowest simulated normal random variables will come from the segment (0.00, 0.25). The lower of the three values in this segment is the ninth one:

$$Min\left[0.540, 0.302, 0.244\right] = 0.244$$

Therefore, we use 0.244 to find the lowest standard normal random variable and the lowest stock price:

$$u_9 = 0.244$$

$$\hat{u}_9 = V_9 = \frac{0.244 + \left[(9-1)\bmod 4\right]}{4} = \frac{0.244 + 0}{4} = 0.06100$$

$$Z_6 = N^{-1}(0.06100) = -1.54643$$

$$S(T) = S_0 e^{(\alpha - \delta - 0.5\sigma^2)T + \sigma z\sqrt{T}} = 10e^{(0.15-0.03-0.5(0.25)^2)1 + 0.25(-1.54643)\sqrt{1}} = 7.4241$$

The highest simulated normal random variable will come from the segment (0.75, 1.00). The highest of the three values in this segment is the eighth one:

$$Max\left[0.450, 0.909, 0.501\right] = 0.909$$

Therefore, we use 0.909 to find the highest standard normal random variable and the highest stock price:

$$u_8 = 0.909$$

$$\hat{u}_8 = V_8 = \frac{0.909 + \left[(8-1)\bmod 4\right]}{4} = \frac{0.909 + 3}{4} = 0.97725$$

$$Z_8 = N^{-1}(0.97725) = 2.00000$$

$$S(T) = S_0 e^{(\alpha - \delta - 0.5\sigma^2)T + \sigma z\sqrt{T}} = 10e^{(0.15-0.03-0.5(0.25)^2)1 + 0.25(2.00000)\sqrt{1}} = 18.0173$$

The difference between the largest and smallest simulated stock prices is:

$$18.0173 - 7.4241 = 10.5932$$

