

## Chapter 24 – Solutions

### Solution 1

**A** Chapter 24, Duration-Hedging

The number of 10-year bonds that must be purchased to duration-hedge the position is:

$$N = -\frac{(T_2 - t) \times P(t, T_2)}{(T_1 - t) \times P(t, T_1)} = -\frac{(3 - 0) \times 85.40}{(10 - 0) \times 73.50} = -0.349$$

Purchasing  $-0.349$  of the 10-year bonds is equivalent to selling 0.349 of the 10-year bonds.

### Solution 2

**C** Chapter 24, Delta-Hedging

The number of 10-year bonds that must be purchased to delta-hedge the position is:

$$N = -\frac{P_r(r, t, T_2)}{P_r(r, t, T_1)} = -\frac{-1.93}{-3.18} = -0.607$$

Purchasing  $-0.607$  of the 10-year bonds is equivalent to selling 0.607 of the 10-year bonds.

### Solution 3

**E** Chapter 24, Rendleman-Bartter Model

In the Rendleman-Bartter model, the short rate follows geometric Brownian motion:

$$dr = ardt + \sigma rdZ \quad \Leftrightarrow \quad d[\ln r] = (a - 0.5\sigma^2)dt + \sigma dZ$$

Choice E is of this form with:

$$a - 0.5\sigma^2 = 0.01 \quad \text{and} \quad \sigma = 0.02$$

### Solution 4

**C** Chapter 24, Vasicek Model

The process describes the Vasicek model with:

$$a = 0.2$$

$$b = 0.1$$

$$\sigma = 0.02$$


The Sharpe ratio is zero, so:

$$\phi = 0$$

We can use these values to find the value that the yield approaches as the time to maturity goes to infinity:

$$\bar{r} = b + \phi \frac{\sigma}{a} - 0.5 \frac{\sigma^2}{a^2} = 0.1 + 0 \left( \frac{0.02}{0.2} \right) - 0.5 \frac{0.02^2}{0.2^2} = 0.095$$

### Solution 5

**B** Chapter 24, Vasicek Model 

The process describes the Vasicek model with:

$$a = 0.2$$

$$b = 0.1$$

$$\sigma = 0.02$$

The Sharpe ratio is zero, so:

$$\phi = 0$$

We can use these values to find the price of the 10-year zero-coupon bond:

$$\bar{r} = b + \phi \frac{\sigma}{a} - 0.5 \frac{\sigma^2}{a^2} = 0.1 + 0 \left( \frac{0.02}{0.2} \right) - 0.5 \frac{0.02^2}{0.2^2} = 0.095$$

$$B(t, T) = \frac{1 - e^{-a(T-t)}}{a} = \frac{1 - e^{-0.2(10)}}{0.2} = 4.32332$$

$$A(t, T) = e^{\bar{r} \times [B(t, T) - (T-t)] - \frac{[B(t, T)]^2 \times \sigma^2}{4a}} = e^{0.095(4.32332 - 10) - \frac{4.32332^2 \times 0.02^2}{4 \times 0.2}} = 0.57774$$

$$P(r, t, T) = A(t, T)e^{-B(t, T)r} = 0.57774e^{-4.32332 \times 0.08} = 0.40881$$

Since the bond matures for \$100, its price is:

$$100 \times 0.40881 = 40.881$$

### Solution 6

**D** Chapter 24, Vasicek Model & Forward Interest Rates 

The process describes the Vasicek model with:

$$a = 0.2$$

$$b = 0.1$$

$$\sigma = 0.02$$

We can use these values to find the price of a 10-year zero-coupon bond:

$$B(0, 10) = \frac{1 - e^{-a(T-t)}}{a} = \frac{1 - e^{-0.2(10)}}{0.2} = 4.32332$$

$$A(0, 10) = 0.57774$$

$$P(r, 0, 10) = A(0, 10)e^{-B(0, 10)r} = 0.57774e^{-4.32332 \times 0.08} = 0.40881$$

We can also find the price of a 9-year zero-coupon bond:

$$B(0,9) = \frac{1 - e^{-a(T-t)}}{a} = \frac{1 - e^{-0.2(9)}}{0.2} = 4.17351$$

$$A(0,9) = 0.62674$$

$$P(r,0,9) = A(0,9)e^{-B(0,9)r} = 0.62674e^{-4.17351 \times 0.08} = 0.44883$$

The forward interest rate, agreed upon now, that applies from time 9 to time 10 can be found with the following formula:

$$R_0(T, T+s) = \frac{1}{P_0(T, T+s)} - 1 = \frac{P(r,0,T)}{P(r,0,T+s)} - 1$$

$$R_0(9,10) = \frac{P(r,0,9)}{P(r,0,10)} - 1 = \frac{0.44883}{0.40881} - 1 = 0.0979$$

Since the amount of the loan is \$100, the payment at the end of 10 years is:

$$100 \times (1 + 0.0979) = 109.79$$

### Solution 7

#### B Chapter 24, Rendleman-Bartter Model



In the Rendleman-Bartter Model, the short-term rate follows geometric Brownian motion. From the Chapter 18 Review Note, we know that the probability that the short rate is greater than 10% at the end of 3 months can be found as:

$$\text{Prob}[r(t) > 0.10] = N(\hat{d}_2)$$


Although we have  $-0.25$  for the volatility parameter in the geometric Brownian motion, we use the absolute value of  $-0.25$  in the formula for  $\hat{d}_2$ , because the  $\sigma$  in the formula for  $\hat{d}_2$  is the standard deviation of the stock's return, and it must therefore be positive:

$$\begin{aligned} \hat{d}_2 &= \frac{\ln\left(\frac{r(t)}{K}\right) + (\alpha - \delta - 0.5\sigma^2)(T-t)}{\sigma\sqrt{T-t}} = \frac{\ln\left(\frac{0.09}{0.10}\right) + (0.10 - 0.5 \times 0.25^2) \times 0.25}{0.25\sqrt{0.25}} \\ &= -0.70538 \end{aligned}$$

The value of  $N(\hat{d}_2)$  is:

$$N(\hat{d}_2) = N(-0.70538) = 0.24029$$

**Solution 8**

A Chapter 24, Cox-Ingersoll-Ross Model 

The process describes the Cox-Ingersoll-Ross model with:

$$a = 0.18$$

$$b = 0.1$$

$$\sigma = 0.17$$

The Sharpe ratio is zero, so  $\bar{\phi}$  is zero:

$$\phi(r, t) = 0$$

$$\frac{\bar{\phi}\sqrt{r}}{\sigma} = 0$$


$$\bar{\phi} = 0$$

We can use these values to find  $\bar{r}$ , the yield as the maturity goes to infinity:

$$\gamma = \sqrt{(a - \bar{\phi})^2 + 2\sigma^2} = \sqrt{(0.18 - 0)^2 + 2 \times 0.17^2} = 0.30033$$

$$\bar{r} = \frac{2ab}{(a - \bar{\phi} + \gamma)} = \frac{2(0.18)(0.10)}{(0.18 - 0 + 0.30033)} = 0.0749$$

**Solution 9**

C Chapter 24, Forward Prices 

We wish to find the price, agreed upon now to be paid in 2 years, for a 1-year bond:


$$P_0(2, 3) = F_{0,2}[P(2, 3)]$$

This price is the ratio of the 3-year bond price to the 2-year bond price:

$$P_0(T, T + s) = \frac{P(0, T + s)}{P(0, T)}$$

$$P_0(2, 3) = \frac{P(0, 3)}{P(0, 2)} = \frac{0.7722}{0.8573} = 0.9007$$

**Solution 10**

C Chapter 24, Black Model 

*This method of presenting forward price volatilities did not appear in the textbook's description of the Black model, but the volatilities are presented this way in Problem 24.2 at the end of the textbook chapter.*

The forward price volatilities are:

$$0.100 = \sqrt{\frac{\text{Var}\{\ln[P_t(1, 2)]\}}{t}} \quad 0.105 = \sqrt{\frac{\text{Var}\{\ln[P_t(2, 3)]\}}{t}} \quad 0.120 = \sqrt{\frac{\text{Var}\{\ln[P_t(3, 4)]\}}{t}}$$

The appropriate volatility for an option that expires in 2 years on a bond that matures in 3 years is:

$$\sigma = 0.105$$

The bond forward price is:

$$F = P_0(T, T+s) = \frac{P(0, T+s)}{P(0, T)} = \frac{P(0, 3)}{P(0, 2)} = 0.9007$$

The values of  $d_1$  and  $d_2$  are:

$$d_1 = \frac{\ln\left(\frac{F}{K}\right) + 0.5\sigma^2 T}{\sigma\sqrt{T}} = \frac{\ln\left(\frac{0.9007}{0.92}\right) + 0.5(0.105)^2(2)}{0.105\sqrt{2}} = -0.06827$$

$$d_2 = d_1 - \sigma\sqrt{T} = -0.06827 - 0.105\sqrt{2} = -0.21676$$

We have:

$$N(d_1) = N(-0.06827) = 0.47279$$

$$N(d_2) = N(-0.21676) = 0.41420$$

The Black formula for the call price is:

$$\begin{aligned} C &= P(0, T)[F \times N(d_1) - K \times N(d_2)] \\ &= 0.8573[0.9007 \times 0.47279 - 0.92 \times 0.41420] \\ &= 0.03840 \end{aligned}$$

### Solution 11

**B** Chapter 24, Black Model



*This method of presenting forward price volatilities did not appear in the textbook's description of the Black model, but the volatilities are presented this way in Problem 24.2 at the end of the textbook chapter.*

The forward price volatilities are:

$$0.100 = \sqrt{\frac{\text{Var}\{\ln[P_t(1, 2)]\}}{t}} \quad 0.105 = \sqrt{\frac{\text{Var}\{\ln[P_t(2, 3)]\}}{t}} \quad 0.120 = \sqrt{\frac{\text{Var}\{\ln[P_t(3, 4)]\}}{t}}$$

The appropriate volatility for an option that expires in 2 years on a bond that matures in 3 years is:

$$\sigma = 0.105$$

The bond forward price is:

$$F = P_0(T, T+s) = \frac{P(0, T+s)}{P(0, T)} = \frac{P(0, 3)}{P(0, 2)} = 0.9007$$

The values of  $d_1$  and  $d_2$  are:

$$d_1 = \frac{\ln\left(\frac{F}{K}\right) + 0.5\sigma^2 T}{\sigma\sqrt{T}} = \frac{\ln\left(\frac{0.9007}{0.92}\right) + 0.5(0.105)^2(2)}{0.105\sqrt{2}} = -0.06827$$

$$d_2 = d_1 - \sigma\sqrt{T} = -0.06827 - 0.105\sqrt{2} = -0.21676$$

We have:

$$N(-d_1) = N(0.06827) = 0.52721$$

$$N(-d_2) = N(0.21676) = 0.58580$$

The Black formula for the put price is:

$$\begin{aligned} P &= P(0, T)[K \times N(-d_2) - F \times N(-d_1)] \\ &= 0.8573[0.92 \times 0.58580 - 0.9007 \times 0.52721] \\ &= 0.05492 \end{aligned}$$

### Solution 12

**D** Chapter 24, Black Model



*This method of presenting forward price volatilities did not appear in the textbook's description of the Black model, but the volatilities are presented this way in Problem 24.2 at the end of the textbook chapter.*

The price of the caplet is:

$$\text{Caplet price} = (1.1075) \times (\text{Put Option Price})$$

The put option:

- has a zero-coupon bond that expires at time 3 as its underlying asset
- expires at time 2
- has a strike price of:

$$K = \frac{1}{1 + K_R} = \frac{1}{1.1075} = 0.9029$$

The forward price volatilities are:

$$0.100 = \sqrt{\frac{\text{Var}\{\ln[P_t(1,2)]\}}{t}} \quad 0.105 = \sqrt{\frac{\text{Var}\{\ln[P_t(2,3)]\}}{t}} \quad 0.120 = \sqrt{\frac{\text{Var}\{\ln[P_t(3,4)]\}}{t}}$$

The appropriate volatility for an option that expires in 2 years on a bond that matures in 3 years is:

$$\sigma = 0.105$$

The bond forward price is:

$$F = P_0(T, T+s) = \frac{P(0, T+s)}{P(0, T)} = \frac{P(0, 3)}{P(0, 2)} = 0.9007$$

The values of  $d_1$  and  $d_2$  are:

$$d_1 = \frac{\ln\left(\frac{F}{K}\right) + 0.5\sigma^2 T}{\sigma\sqrt{T}} = \frac{\ln\left(\frac{0.9007}{0.9029}\right) + 0.5(0.105)^2(2)}{0.105\sqrt{2}} = 0.05782$$

$$d_2 = d_1 - \sigma\sqrt{T} = 0.05782 - 0.105\sqrt{2} = -0.09067$$

We have:

$$N(-d_1) = N(-0.05782) = 0.47695$$

$$N(-d_2) = N(0.09067) = 0.53612$$

The Black formula for the put price is:

$$P = P(0, T)[K \times N(-d_2) - F \times N(-d_1)]$$

$$= 0.8573[0.9029 \times 0.53612 - 0.9007 \times 0.47695]$$

$$= 0.04670$$

The price of the caplet is:

$$\text{Caplet price} = (1.1075) \times (\text{Put Option Price}) = 1.1075 \times 0.04670 = 0.05172$$

### Solution 13

**C** Chapter 24, Black Model



*This method of presenting forward price volatilities did not appear in the textbook's description of the Black model, but the volatilities are presented this way in Problem 24.2 at the end of the textbook chapter.*

The price of the caplet is:

$$\text{Caplet price} = (1.12) \times (\text{Put Option Price})$$

The put option:

- has a zero-coupon bond that expires at time 4 as its underlying asset
- expires at time 3
- has a strike price of:

$$K = \frac{1}{1 + K_R} = \frac{1}{1.12} = 0.8929$$

The forward price volatilities are:

$$0.100 = \sqrt{\frac{\text{Var}\{\ln[P_t(1,2)]\}}{t}} \quad 0.105 = \sqrt{\frac{\text{Var}\{\ln[P_t(2,3)]\}}{t}} \quad 0.120 = \sqrt{\frac{\text{Var}\{\ln[P_t(3,4)]\}}{t}}$$

The appropriate volatility for an option that expires in 3 years on a bond that matures in 4 years is:

$$\sigma = 0.12$$

The bond forward price is:

$$F = P_0(T, T+s) = \frac{P(0, T+s)}{P(0, T)} = \frac{P(0, 4)}{P(0, 3)} = \frac{0.6956}{0.7722} = 0.9008$$

The values of  $d_1$  and  $d_2$  are:

$$d_1 = \frac{\ln\left(\frac{F}{K}\right) + 0.5\sigma^2 T}{\sigma\sqrt{T}} = \frac{\ln\left(\frac{0.9008}{0.8929}\right) + 0.5(0.12)^2(3)}{0.12\sqrt{3}} = 0.14655$$

$$d_2 = d_1 - \sigma\sqrt{T} = 0.14655 - 0.12\sqrt{3} = -0.06130$$

We have:

$$N(-d_1) = N(-0.14655) = 0.44174$$

$$N(-d_2) = N(0.06130) = 0.52444$$

The Black formula for the put price is:

$$\begin{aligned} P &= P(0, T)[K \times N(-d_2) - F \times N(-d_1)] \\ &= 0.7722[0.8929 \times 0.52444 - 0.9008 \times 0.44174] \\ &= 0.054308 \end{aligned}$$

The price of the caplet is:

$$\text{Caplet price} = (1.12) \times (\text{Put Option Price}) = 1.12 \times 0.054308 = 0.060825$$

### Solution 14

**E** Chapter 24, Floorlet in the Black Model ■■■■■

*This method of presenting forward price volatilities did not appear in the textbook's description of the Black model, but the volatilities are presented this way in Problem 24.2 at the end of the textbook chapter.*

The price of the floorlet is:

$$\text{Floorlet price} = (1.12) \times (\text{Call Option Price})$$

The call option:

- has a zero-coupon bond that expires at time 4 as its underlying asset
- expires at time 3
- has a strike price of:

$$K = \frac{1}{1 + K_R} = \frac{1}{1.12} = 0.8929$$

The forward price volatilities are:

$$0.100 = \sqrt{\frac{\text{Var}\{\ln[P_t(1,2)]\}}{t}} \quad 0.105 = \sqrt{\frac{\text{Var}\{\ln[P_t(2,3)]\}}{t}} \quad 0.120 = \sqrt{\frac{\text{Var}\{\ln[P_t(3,4)]\}}{t}}$$

The appropriate volatility for an option that expires in 3 years on a bond that matures in 4 years is:

$$\sigma = 0.12$$

The bond forward price is:

$$F = P_0(T, T+s) = \frac{P(0, T+s)}{P(0, T)} = \frac{P(0, 4)}{P(0, 3)} = \frac{0.6956}{0.7722} = 0.9008$$

The values of  $d_1$  and  $d_2$  are:

$$d_1 = \frac{\ln\left(\frac{F}{K}\right) + 0.5\sigma^2 T}{\sigma\sqrt{T}} = \frac{\ln\left(\frac{0.9008}{0.8929}\right) + 0.5(0.12)^2(3)}{0.12\sqrt{3}} = 0.14655$$

$$d_2 = d_1 - \sigma\sqrt{T} = 0.14655 - 0.12\sqrt{3} = -0.06130$$

We have:

$$N(d_1) = N(0.14655) = 0.55826$$

$$N(d_2) = N(-0.06130) = 0.47556$$

The Black formula for the call price is:

$$\begin{aligned} C &= P(0, T)[F \times N(d_1) - K \times N(d_2)] \\ &= 0.7722[0.9008 \times 0.55826 - 0.8929 \times 0.47556] \\ &= 0.060444 \end{aligned}$$

The price of a floorlet with a notional amount of \$1 is:

$$\text{Floorlet price} = (1.12) \times (\text{Call Option Price}) = 1.12 \times 0.060444 = 0.067697$$

Since the notional amount is \$1,000, we multiply by \$1,000 to obtain the answer:

$$\$1,000 \times 0.067697 = 67.697$$

**Solution 15**

**E** Chapter 24, Forward Rate Agreements 

By purchasing the caplet and selling the floorlet, the investor has created a forward rate agreement that locks in the rate  $K_R$  from time 2 to time 3. The investor's time 3 cash flow is the payment from the caplet minus the payment from the floorlet:

$$\text{Max}[0, R_2 - K_R] - \text{Max}[0, K_R - R_2] = R_2 - K_R$$

A forward rate agreement, which can be entered into at zero cost, provides a cash flow at time 3 of:

$$R_2 - R_0(2, 3)$$


Both  $R_0(2, 3)$  and  $K_R$  are known values at time 0. Since the cost of both strategies is the same (zero), to preclude arbitrage, we must have:

$$K_R = R_0(2, 3)$$

We can find the forward rate that applies from time 2 to time 3:

$$K_R = R_0(2, 3) = \frac{P(0, 2)}{P(0, 3)} - 1 = \frac{0.8573}{0.7722} - 1 = 0.1102$$


**Solution 16**

**C** Chapter 24, Binomial Interest Rate Model 

The price of the 2-year zero-coupon bond is:

$$P(0, 2) = E^* \left[ e^{-\sum_{i=0}^1 r_i} \right] = 0.5 \times e^{-(0.12+0.16)} + 0.5 \times e^{-(0.12+0.08)} = 0.7873$$

**Solution 17**

**A** Chapter 24, Binomial Interest Rate Model 

The price of the 3-year zero-coupon bond is:

$$\begin{aligned} P_0(0, 3) &= E^* \left[ e^{-\sum_{i=0}^2 r_i} \right] \\ &= 0.25 \times e^{-(0.12+0.16+0.20)} \\ &\quad + 0.25 \times e^{-(0.12+0.16+0.12)} \\ &\quad + 0.25 \times e^{-(0.12+0.08+0.12)} \\ &\quad + 0.25 \times e^{-(0.12+0.08+0.04)} \\ &= 0.7005 \end{aligned}$$

**Solution 18****B** Chapter 24, Binomial Interest Rate Model

The terminology can be a bit confusing. This same option could also be described as a 2-year put on a 3-year zero-coupon bond, with the understanding that the 3-year bond will be a 1-year bond at the end of 2 years. In this case, since a 1-year bond will have matured by the end of 2 years, it is apparent that the 1-year bond in the question refers to a 1-year bond at the end of 2 years.

The tree of prices for the 3-year bond is:

|        |        |        |
|--------|--------|--------|
|        | 0.8187 | 1.0000 |
| 0.7267 |        |        |
|        | 0.8869 | 1.0000 |
| 0.7005 |        |        |
|        | 0.8869 | 1.0000 |
| 0.8528 |        |        |
|        | 0.9608 | 1.0000 |

There is no need to calculate the first two columns of the tree to answer this question, but they are shown above for illustrative purposes.

The third column is easily obtained by discounting a 1-year bond at the possible interest rates at time 2:

$$e^{-0.20} = 0.8187$$

$$e^{-0.12} = 0.8869$$

$$e^{-0.04} = 0.9608$$

The put option pays off only if the price falls to \$0.8187. If this occurs, the option payoff is:

$$V_2 = 0.85 - 0.8187 = 0.0313$$

There is a 0.25 risk-neutral probability of this occurring, so the price of the put option is:

$$V_0 = E^* \left[ V_2 \times e^{-\sum_{i=0}^{2-1} r_i} \right] = 0.25(0.0313)e^{-(0.12+0.16)} = 0.0059$$

**Solution 19**

**C** Chapter 24, Binomial Interest Rate Model



The tree of prices for the 3-year bond is:

|        |        |        |
|--------|--------|--------|
|        | 0.9048 | 1.0000 |
| 0.8523 |        |        |
|        | 0.9418 | 1.0000 |
| 0.8361 |        |        |
|        | 0.9418 | 1.0000 |
| 0.9233 |        |        |
|        | 0.9802 | 1.0000 |

*There is no need to calculate the first two columns of the tree to answer this question, but they are shown above for illustrative purposes.*

The third column is easily obtained by discounting a 1-year bond at the possible interest rates at time 2:

$$e^{-0.10} = 0.9048$$

$$e^{-0.06} = 0.9418$$

$$e^{-0.02} = 0.9802$$

The call option pays off if the final interest rate is 0.06 or lower. The possible payoffs are:

$$r_2 = 10\% \quad \Rightarrow \quad V_2 = 0.0000$$

$$r_2 = 6\% \quad \Rightarrow \quad V_2 = 0.9418 - 0.92 = 0.0218$$

$$r_2 = 2\% \quad \Rightarrow \quad V_2 = 0.9802 - 0.92 = 0.0602$$

Each of the 4 paths has a 0.25 risk-neutral probability of occurring. This leads to the following call price:

$$\begin{aligned} V_0 &= E^* \left[ V_2 \times e^{-\sum_{i=0}^{2-1} r_i} \right] \\ &= 0.25(0.0000)e^{-(0.06+0.08)} \\ &\quad + 0.25(0.0218)e^{-(0.06+0.08)} \\ &\quad + 0.25(0.0218)e^{-(0.06+0.04)} \\ &\quad + 0.25(0.0602)e^{-(0.06+0.04)} \\ &= 0.02327 \end{aligned}$$

**Solution 20****B** Chapter 24, Black-Derman-Toy Model *The amount of the loan does not affect the forward rate.*

The price of a 3-year zero-coupon bond is:


$$\begin{aligned}
 P_0(0,3) &= E^* \left[ \prod_{i=0}^2 \frac{1}{(1+r_i)} \right] \\
 &= 0.25 \times \frac{1}{1.10} \times \frac{1}{1.1517} \times \frac{1}{1.1889} \\
 &\quad + 0.25 \times \frac{1}{1.10} \times \frac{1}{1.1517} \times \frac{1}{1.1489} \\
 &\quad + 0.25 \times \frac{1}{1.10} \times \frac{1}{1.1293} \times \frac{1}{1.1489} \\
 &\quad + 0.25 \times \frac{1}{1.10} \times \frac{1}{1.1293} \times \frac{1}{1.1173} \\
 &= 0.6930
 \end{aligned}$$

The price of a 2-year zero-coupon bond is:

$$\begin{aligned}
 P_0(0,2) &= E^* \left[ \prod_{i=0}^1 \frac{1}{(1+r_i)} \right] \\
 &= 0.5 \times \frac{1}{1.10} \times \frac{1}{1.1517} + 0.5 \times \frac{1}{1.10} \times \frac{1}{1.1293} \\
 &= 0.7972
 \end{aligned}$$

The contract rate on the forward rate agreement is the forward rate:

$$R_0(2,3) = \frac{P(0,2)}{P(0,3)} - 1 = \frac{0.7972}{0.6930} - 1 = 0.1503$$

**Solution 21****A** Chapter 24, Risk-Neutral Vasicek Model 

Greg uses the Vasicek model for the short rate. The risk-neutral version of Greg's model is:

$$\begin{aligned}
 dr &= [a(r) + \sigma(r)\phi(r,t)]dt + \sigma(r)d\tilde{Z} \\
 &= [a(b-r) + \sigma\phi]dt + \sigma d\tilde{Z} \\
 &= [0.2(0.1-r) + 0.05 \times 0.08]dt + 0.05d\tilde{Z} \\
 &= 0.2(0.12-r)dt + 0.05d\tilde{Z}
 \end{aligned}$$

Colleen's model will produce the same price as Greg's if the risk-neutral version of her model is the same as the risk-neutral version of Greg's model.

The risk-neutral version of the model shown in Choice A is:

$$\begin{aligned} dr &= [a(b-r) + \sigma\phi]dt + \sigma d\tilde{Z} = [0.2(0.08-r) + 0.05 \times 0.16]dt + 0.05d\tilde{Z} \\ &= 0.2(0.12-r)dt + 0.05d\tilde{Z} \end{aligned}$$

Therefore, the risk-neutral version of Choice A is the same as the risk-neutral version of Greg's model, and Choice A is correct.

For the sake of thoroughness, let's consider the other choices as well. As shown below, the risk-neutral version of Choice B does not match the risk-neutral version of Greg's model:

$$\begin{aligned} dr &= [a(b-r) + \sigma\phi]dt + \sigma d\tilde{Z} = [0.2(0.08-r) + 0.05 \times 0.32]dt + 0.05d\tilde{Z} \\ &= 0.2(0.16-r)dt + 0.05d\tilde{Z} \end{aligned}$$

As shown below, the risk-neutral version of Choice C does not match the risk-neutral version of Greg's model:

$$\begin{aligned} dr &= [a(b-r) + \sigma\phi]dt + \sigma d\tilde{Z} = [0.2(0.12-r) + 0.05 \times 0.08]dt + 0.05d\tilde{Z} \\ &= 0.2(0.14-r)dt + 0.05d\tilde{Z} \end{aligned}$$

Choices D and E are unlikely to be correct, because they are based on the Cox-Ingersoll-Ross model. Since we have already shown that Choice A is correct, and the question tells us that only one model produces the same price as Greg's model, Choices D and E must be incorrect.

## Solution 22

**B** Chapter 24, Black-Derman-Toy Model



Although the cap payments are made at the end of each year, we will value them at the beginning of each year using the following formula:

$$T\text{-year caplet payoff at time } (T-1) = \frac{\text{Max}[0, R_{T-1} - K_R]}{1 + R_{T-1}} \times \text{Notional}$$

A 2-year cap consists of a 1-year caplet and a 2-year caplet.

The payoff for the 1-year caplet is zero since 10% is less than 14%.

The payoff for the 2-year caplet is positive only when the short-term interest rate increases to 15.17%:

$$\begin{aligned} \text{2-year caplet payoff at time 1} &= \frac{\text{Max}[0, R_1 - K_R]}{1 + R_1} \times 100 = \frac{0.1517 - 0.14}{1.1517} \times 100 \\ &= 1.0159 \end{aligned}$$

This payment is made at time 1 and it has a 50% probability of occurring, so the value of the 2-year cap is:

$$\frac{0.5 \times 1.0159}{1.10} = 0.462$$

### Solution 23

C Chapter 24, Black-Derman-Toy Model ■ ■ ■ ■ □

Although the cap payments are made at the end of each year, we will value them at the beginning of each year using the following formula:

$$T\text{-year caplet payoff at time } (T - 1) = \frac{\text{Max}[0, R_{T-1} - K_R]}{1 + R_{T-1}} \times \text{Notional}$$

A 3-year cap consists of a 1-year caplet, a 2-year caplet, and a 3-year caplet.

The payoff for the 1-year caplet is zero since 10% is less than 14%.

The payoff for the 2-year caplet is positive only when the short-term interest rate increases to 15.17%:

$$R_1 = 0.1517 \Rightarrow \frac{0.1517 - 0.14}{1.1517} \times 100 = 1.0159$$

The payoff for the 3-year caplet is positive when the short-term rate exceeds 14%:

$$R_2 = 0.1889 \Rightarrow \frac{0.1889 - 0.14}{1.1889} \times 100 = 4.1130$$

$$R_2 = 0.1489 \Rightarrow \frac{0.1489 - 0.14}{1.1489} \times 100 = 0.7747$$

The possible payments from the cap are illustrated in the tree below:

$$\begin{array}{r} 4.1130 \\ 1.0159 \\ 0.0000 \quad 0.7747 \\ 0.0000 \\ 0.0000 \end{array}$$

The value of the 2-year caplet is:

$$\text{Value of 2-year caplet} = \frac{0.5 \times 1.0159}{1.10} = 0.4618$$

The value of the 3-year caplet is:

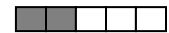
$$\begin{aligned} \text{Value of the 3-year caplet} &= 0.25 \times \frac{1}{1.10} \times \frac{1}{1.1517} \times 4.1130 \\ &\quad + 0.25 \times \frac{1}{1.10} \times \frac{1}{1.1517} \times 0.7747 \\ &\quad + 0.25 \times \frac{1}{1.10} \times \frac{1}{1.1293} \times 0.7747 \\ &= 1.1204 \end{aligned}$$

The value of the 3-year cap is:

$$\begin{aligned} \text{Value of 3-year cap} &= (\text{1-year caplet}) + (\text{2-year caplet}) + (\text{3-year caplet}) \\ &= 0.000 + 0.4618 + 1.1204 = 1.5822 \end{aligned}$$

### Solution 24

A Chapter 24, Black-Derman-Toy Model



There are two ways to answer this question.

#### Method 1

First, we consider the quick way. The short-term rates in one period are  $R_1$  and  $e^{2\sigma_1\sqrt{h}} R_1$ :

$$\begin{aligned} R_1 &= 0.1293 \\ e^{2\sigma_1\sqrt{h}} R_1 &= 0.1517 \end{aligned}$$

We can solve for  $\sigma_1$ , which is the yield volatility for the 2-year bond:

$$\begin{aligned} \frac{e^{2\sigma_1\sqrt{h}} R_1}{R_1} &= \frac{0.1517}{0.1293} \\ \sigma_1 &= \ln\left(\frac{0.1517}{0.1293}\right) \times 0.5 = 0.0799 \end{aligned}$$

*This method only works for the 2-year bond! For longer term bonds, we must use the method shown below.*

#### Method 2

The second method takes a bit longer. First, we create the tree of prices for the 2-year bond:

$$\begin{array}{r} 0.8683 \\ 0.7972 \\ 0.8855 \end{array}$$

The calculations to obtain these prices are:

$$P(1, 2, r_u) = \frac{1}{1.1517} = 0.8683$$

$$P(1, 2, r_d) = \frac{1}{1.1293} = 0.8855$$

$$P(0, 2) = \frac{1}{1.10} \times 0.5(0.8683 + 0.8855) = 0.7972$$

We don't need to calculate  $P(0, 2)$  to answer this question, but we included it for completeness.

The formula for the one-year yield volatility for a  $T$ -year zero-coupon bond is:

$$\text{Yield volatility} = 0.5 \times \ln \left[ \frac{P(1, T, r_u)^{-1/(T-1)} - 1}{P(1, T, r_d)^{-1/(T-1)} - 1} \right]$$

The yield volatility of the 2-year bond is:

$$\text{Yield volatility} = 0.5 \times \ln \left[ \frac{0.8683^{-1/(2-1)} - 1}{0.8855^{-1/(2-1)} - 1} \right] = 0.5 \times \ln \left( \frac{0.1517}{0.1293} \right) = 0.0799$$

### Solution 25

**B** Chapter 24, Black-Derman-Toy Model



The tree of prices for the 3-year bond is:

$$\begin{array}{ccc} & & 0.8411 \\ & & 0.7430 \\ 0.6930 & & 0.8704 \\ & & 0.7816 \\ & & 0.8950 \end{array}$$

The calculations to obtain these prices are:

$$P(2, 3, r_{uu}) = \frac{1}{1.1889} = 0.8411$$

$$P(2, 3, r_{ud}) = P(2, 3, r_{du}) = \frac{1}{1.1489} = 0.8704$$

$$P(2, 3, r_{dd}) = \frac{1}{1.1173} = 0.8950$$

$$P(1, 3, r_u) = \frac{0.5(0.8411 + 0.8704)}{1.1517} = 0.7430$$

$$P(1, 3, r_d) = \frac{0.5(0.8704 + 0.8950)}{1.1293} = 0.7816$$

$$P(0, 3) = \frac{0.5(0.7430 + 0.7816)}{1.10} = 0.6930$$

We don't need to calculate  $P(0,3)$  to answer this question, but we included it for completeness.

The formula for the one-year yield volatility for a  $T$ -year zero-coupon bond is:

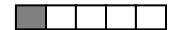
$$\text{Yield volatility} = 0.5 \times \ln \left[ \frac{P(1, T, r_u)^{-1/(T-1)} - 1}{P(1, T, r_d)^{-1/(T-1)} - 1} \right]$$

The yield volatility of the 3-year bond is:

$$\text{Yield volatility} = 0.5 \times \ln \left[ \frac{0.7430^{-1/(3-1)} - 1}{0.7816^{-1/(3-1)} - 1} \right] = 0.100$$

### Solution 26

**D** Chapter 24, Vasicek Model



This question is based on Table 24.1 in the textbook.

The process can be written as:

$$dr = 0.25(0.10 - r)dt + 0.01dZ$$

This is the Vasicek model with:

$$a = 0.25$$

$$b = 0.10$$

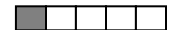
$$\sigma = 0.01$$

The gap between  $r$  and  $b$  is expected to close at a rate equal to  $a$  times the gap:

$$X = 0.25(0.10 - 0.05) = 0.0125$$

### Solution 27

**E** Chapter 24, Continuous-Time Interest Rate Models



The page number references below apply to the Second Edition, First Printing of Derivatives Markets.

Choice A is true, because the Rendleman-Bartter model assumes that the short-rate follows geometric Brownian motion. See the sentence preceding Equation 24.24 on page 785.


Choice B is true, because a shortcoming of the Vasicek model is that interest rates can be negative. See the last sentence of the third paragraph on page 786.

Choice C is true, because a shortcoming of the Vasicek model is that the variance does not change with the short-rate. See the last sentence of the third paragraph on page 786.

Choice D is true, because in the CIR model the variance increases with the short-rate. See the second bullet point on the lower half of page 787.

Choice E is false, because the Rendleman-Bartter model is not mean-reverting. See the last paragraph on page 785.

### Solution 28

**E** Chapter 24, Vasicek Model 

As noted in the Chapter 24 Review Note, the textbook contains a typo in the formula for  $A(t, T)$  when  $a = 0$ . Since this typo was not noted in the textbook's errata at press time, we use the incorrect version of the formula (as it appears in the textbook) below. The correct price is actually \$68.44.

The process describes the Vasicek model with:

$$\begin{aligned} a &= 0 \\ \sigma &= 0.1 \end{aligned}$$

The value of  $b$  is unknown, but it has no impact on the model since  $a$  is zero.

The Sharpe ratio is:

$$\phi = 0.15$$

We can use these values to find the price of the 5-year zero-coupon bond. When  $a = 0$ , we have:

$$\begin{aligned} B(0, T) &= T = 5 \\ A(0, T) &= e^{0.5\sigma\phi T^2 + \frac{\sigma^2 T^3}{6}} = e^{0.5(0.1)(0.15)(5)^2 + \frac{(0.1)^2(5)^3}{6}} = 1.485622 \\ P(r, 0, T) &= A(0, T)e^{-B(0, T)r} = 1.485622e^{-5 \times 0.08} = 0.995842 \end{aligned}$$

Since the bond matures for \$100, its price is:

$$100 \times 0.995842 = 99.5842$$

### Solution 29

**D** Chapter 24, Black-Derman-Toy Model 

The value of  $r_0$  is the same as the yield on the 1-year zero-coupon bond, and the yield volatility of the 2-year bond is  $\sigma_1$ :

$$\begin{aligned} r_0 &= 8\% \\ \sigma_1 &= 12\% \end{aligned}$$

The first part of the interest rate tree is:

$$\begin{array}{ccc} r_u = R_1 e^{2\sigma_1} & & R_1 e^{0.24} \\ r_0 & \Rightarrow & 8\% \\ r_d = R_1 & & R_1 \end{array}$$

To obtain the value of  $r_u$ , we need to determine the value of  $R_1$ . The tree must correctly price the 2-year zero-coupon bond, so:

$$P(0,2) = \frac{1}{1.08} (0.5) \left( \frac{1}{1 + R_1 e^{0.24}} + \frac{1}{1 + R_1} \right)$$

$$\left( \frac{1}{1.09} \right)^2 = \frac{1}{1.08} (0.5) \left( \frac{1}{1 + R_1 e^{0.24}} + \frac{1}{1 + R_1} \right)$$

$$1.818028 = \frac{1}{1 + R_1 e^{0.24}} + \frac{1}{1 + R_1}$$

There are 2 ways to answer this question.

### Method 1

Since we are provided with the possible answers, a perfectly valid approach during the exam is to use trial & error to see which one works. We use the following relationships:

$$R_1 e^{0.24} = r_u \quad \Rightarrow \quad R_1 = r_u e^{-0.24}$$

Let's try Choice C first:

$$\frac{1}{1 + 0.1017e^{-0.24}} + \frac{1}{1.1017} = 1.833614$$

The value 1.833614 is greater than the correct answer of 1.818028, so let's try a higher interest rate. Let's try Choice D:

$$\frac{1}{1 + 0.1122e^{-0.24}} + \frac{1}{1.1122} = 1.818017$$

The value of 1.818017 is within rounding tolerance to the correct answer of 1.818028, so Choice D is the correct answer.

### Method 2

The second method is more time consuming:

$$1.818028 = \frac{1}{1 + R_1 e^{0.24}} + \frac{1}{1 + R_1}$$

$$1.818028(1 + R_1 e^{0.24})(1 + R_1) = 1 + R_1 + 1 + R_1 e^{0.24}$$

$$1.818028(1 + R_1 e^{0.24} + R_1 + R_1^2 e^{0.24}) = 2 + 2.271249R_1$$

$$1.818028 + 4.129196R_1 + 2.311168R_1^2 = 2 + 2.271249R_1$$

$$2.311168R_1^2 + 1.857947R_1 - 0.181971 = 0$$

We can use the quadratic formula to find  $R_1$  :

$$R_1 = \frac{-1.857947 \pm \sqrt{1.857947^2 - 4(2.311168)(-0.181971)}}{2(2.311168)}$$

$$R_1 = 0.088253 \quad \text{or} \quad R_1 = -0.892153$$

We use the positive value of  $R_1$  since the negative value doesn't make sense for interest rates. The value of  $r_u$  is:

$$r_u = R_1 e^{2\sigma_1} = 0.088253 e^{2 \times 0.12} = 0.112192$$

### Solution 30

**A** Chapter 24, Black-Derman-Toy Model ████████

The value of  $r_0$  is the same as the yield on the 1-year zero-coupon bond, and the yield volatility of the 2-year bond is  $\sigma_1$  :

$$r_0 = 8\%$$

$$\sigma_1 = 12\%$$

The first part of the interest rate tree is:

$$\begin{array}{ccc} r_u = R_1 e^{2\sigma_1} & & R_1 e^{0.24} \\ r_0 & \Rightarrow & 8\% \\ r_d = R_1 & & R_1 \end{array}$$

To obtain the value of  $r_u$ , we need to determine the value of  $R_1$ . The tree must correctly price the 2-year zero-coupon bond, so:

$$P(0,2) = \frac{1}{1.08} (0.5) \left( \frac{1}{1 + R_1 e^{0.24}} + \frac{1}{1 + R_1} \right)$$

$$\left( \frac{1}{1.09} \right)^2 = \frac{1}{1.08} (0.5) \left( \frac{1}{1 + R_1 e^{0.24}} + \frac{1}{1 + R_1} \right)$$

$$1.818028 = \frac{1}{1 + R_1 e^{0.24}} + \frac{1}{1 + R_1}$$

$$1.818028 (1 + R_1 e^{0.24}) (1 + R_1) = 1 + R_1 + 1 + R_1 e^{0.24}$$

$$1.818028 (1 + R_1 e^{0.24} + R_1 + R_1^2 e^{0.24}) = 2 + 2.271249 R_1$$

$$1.818028 + 4.129196 R_1 + 2.311168 R_1^2 = 2 + 2.271249 R_1$$

$$2.311168 R_1^2 + 1.857947 R_1 - 0.181971 = 0$$

We can use the quadratic formula to find  $R_1$  :

$$R_1 = \frac{-1.857947 \pm \sqrt{1.857947^2 - 4(2.311168)(-0.181971)}}{2(2.311168)}$$

$$R_1 = 0.088253 \quad \text{or} \quad R_1 = -0.892153$$

We use the positive value of  $R_1$  . The values of  $r_u$  and  $r_d$  are:

$$r_u = R_1 e^{2\sigma_1} = 0.088253 e^{2 \times 0.12} = 0.112192$$

$$r_d = R_1 = 0.088253$$

The first part of the interest rate tree is:

$$\begin{array}{c} 11.2192\% \\ 8.0000\% \\ 8.8253\% \end{array}$$

The cap pays off only if the interest rate reaches 11.2192%. There is a 50% probability of this occurring, so the value of the interest rate cap is:

$$0.5 \times \frac{100 \times (0.112192 - 0.1)}{1.08 \times 1.112192} = 0.5075$$

### Solution 31

**B** Chapter 24, Black-Derman-Toy Model



The value of  $r_0$  is the same as the yield on the 1-year zero-coupon bond, so  $r_0 = 7\%$  . The first part of the interest rate tree is:

$$\begin{array}{c} r_u = 0.076821 e^{2\sigma_1} \\ 7\% \\ r_d = 0.076821 \end{array}$$

The value of  $\sigma_1$  must correctly price the 2-year zero-coupon bond, so:

$$P(0,2) = \frac{1}{1.07} (0.5) \left( \frac{1}{1 + 0.076821e^{2\sigma_1}} + \frac{1}{1 + 0.076821} \right)$$

$$\left( \frac{1}{1.08} \right)^2 = \frac{1}{1.07} (0.5) \left( \frac{1}{1 + 0.076821e^{2\sigma_1}} + \frac{1}{1 + 0.076821} \right)$$

$$1.834705(1 + 0.076821e^{2\sigma_1})(1 + 0.076821) = 1 + 0.076821 + 1 + 0.076821e^{2\sigma_1}$$

$$1.834705(1.076821 + 0.082722e^{2\sigma_1}) = 2.076821 + 0.076821e^{2\sigma_1}$$

$$0.074950e^{2\sigma_1} = 0.101172$$

$$2\sigma_1 = \ln(1.349855)$$

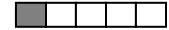
$$\sigma_1 = 0.149998$$

The yield volatility of the 2-year zero-coupon bond is:

$$\sigma_1 = 0.149998$$

### Solution 32

**D** Chapter 24, Black-Derman-Toy Model



Each node is  $e^{2\sigma_i\sqrt{h}}$  times the one below it. This means that 13.51% is  $e^{4\sigma_2}$  times as large as 7.17%:

$$0.1351 = 0.0717e^{4\sigma_2}$$

$$\sigma_2 = \ln\left(\frac{0.1351}{0.0717}\right) \times \frac{1}{4} = 0.15838$$

The interest rate for the node above 7.17% is:

$$X = 0.0717e^{2\sigma_2} = 0.0717e^{2 \times 0.15838} = 0.09842$$

### Solution 33

**E** Chapter 24, Duration-Hedging



*This question is similar to Problem 24.4 on at the end of the textbook chapter.*

*Let's assume that the bonds both mature for \$1. We'll scale up the result at the end.*

The prices of the bonds are:

$$P(0,2) = \frac{1}{e^{2(0.08)}} = 0.85214379$$

$$P(0,7) = \frac{1}{e^{7(0.08)}} = 0.57120906$$

The quantity of 7-year bonds that must be purchased is:

$$N = -\frac{(T_2 - t) \times P(t, T_2)}{(T_1 - t) \times P(t, T_1)} = -\frac{2 \times P(0, 2)}{7 \times P(0, 7)} = -\frac{2 \times e^{-2(0.08)}}{7 \times e^{-7(0.08)}} = -0.426236$$

Since  $N$  is negative, 0.426236 7-year bonds are sold.

The amount that is lent is:

$$\begin{aligned} W &= -P(t, T_2) - N \times P(t, T_1) = -0.85214379 + 0.426236 \times 0.57120906 \\ &= -0.60867413 \end{aligned}$$

Since  $W$  is negative, 0.60867413 is borrowed at the short-term rate of 8%.

This hedged position has an initial value of zero. After 1 day, the short-term rate increases to 8.5%, and the new value of the position is:

$$e^{-\left(2 - \frac{1}{365}\right)0.085} - 0.426236e^{-\left(7 - \frac{1}{365}\right)0.085} - 0.60867413e^{\left(\frac{1}{365}\right)0.08} = -0.00009662$$

Let's scale this up by \$1,000,000:

$$1,000,000 \times (-0.00009662) = -96.62$$

The investor loses \$96.62 on the hedge.

### Solution 34

**C** Chapter 24, Risk-Neutral Vasicek Model



The short-rate follows the Vasicek model:

$$dr = a(b - r)dt + \sigma dZ$$

From the realistic process provided in the question, we observe that  $b = 0.12$ :

$$dr = a(0.12 - r)dt + \sigma dZ \quad \Rightarrow \quad b = 0.12$$

The risk-neutral process is given by:

$$dr = [a(r) + \sigma(r)\phi(r, t)]dt + \sigma(r)d\tilde{Z} = [a(b - r) + \sigma\phi]dt + \sigma d\tilde{Z}$$

From risk-neutral process provided in the question, we observe that  $\sigma = 0.03$  and  $a = 0.3$ :

$$dr = 0.3(0.125 - r)dt + 0.03d\tilde{Z} \quad \Rightarrow \quad \sigma = 0.03 \text{ and } a = 0.3$$

We use the coefficient of  $dt$  in the risk-neutral process to determine the Sharpe ratio  $\phi$ :

$$\begin{aligned} a(b-r) + \sigma\phi &= 0.3(0.125-r) \\ 0.3(0.12-r) + 0.03\phi &= 0.3(0.125-r) \\ 0.12-r + 0.1\phi &= 0.125-r \\ 0.12 + 0.1\phi &= 0.125 \\ \phi &= \frac{0.125-0.12}{0.1} \\ \phi &= 0.05 \end{aligned}$$

The values of  $B(0,10)$  and  $q(0.08,0,10)$  are:

$$\begin{aligned} B(t,T) &= \frac{1-e^{-a(T-t)}}{a} = \frac{1-e^{-0.3 \times 10}}{0.3} = 3.16738 \\ q(0.08,0,10) &= B(0,10)\sigma(r) = B(0,10)\sigma = 3.16738 \times 0.03 = 0.09502 \end{aligned}$$

We can now determine the expected return on the bond:

$$\begin{aligned} \phi(r,t) &= \frac{\alpha(r,t,T) - r}{q(r,t,T)} \\ 0.05 &= \frac{\alpha(0.08,0,10) - 0.08}{0.09502} \\ \alpha(0.08,0,10) &= 0.08475 \end{aligned}$$

### Solution 35

**D** Chapter 24, Risk-Neutral Cox-Ingersoll-Ross Model



The short-rate follows the Cox-Ingersoll-Ross model:

$$dr = a(b-r)dt + \sigma\sqrt{r}dZ$$

From the realistic process provided in the question, we observe that  $b = 0.08$ :

$$dr = a(0.08-r)dt + \sigma\sqrt{r}dZ \Rightarrow b = 0.08$$

The risk-neutral process is given by:

$$\begin{aligned} dr &= [a(r) + \sigma(r)\phi(r,t)]dt + \sigma(r)d\tilde{Z} \\ &= \left[ a(b-r) + \sigma\sqrt{r}\bar{\phi} \frac{\sqrt{r}}{\sigma} \right] dt + \sigma\sqrt{r}d\tilde{Z} \\ &= [a(b-r) + r\bar{\phi}]dt + \sigma\sqrt{r}d\tilde{Z} \end{aligned}$$

From risk-neutral process provided in the question, we observe that  $\sigma = 0.04$ :

$$dr = 0.2(0.096-r)dt + 0.04\sqrt{r}d\tilde{Z} \Rightarrow \sigma = 0.04$$

We use the coefficient of  $dt$  in the risk-neutral process to determine  $a$  and  $\bar{\phi}$ :

$$a(b - r) + r\bar{\phi} = 0.2(0.096 - r)$$

$$ab - (a - \bar{\phi})r = 0.0192 - 0.2r$$

$$0.08a - (a - \bar{\phi})r = 0.0192 - 0.2r$$

From the equation above, we observe that:

$$0.08a = 0.0192 \quad \Rightarrow \quad a = 0.24$$

$$a - \bar{\phi} = 0.2 \quad \Rightarrow \quad \bar{\phi} = a - 0.2 = 0.24 - 0.2 = 0.04$$

In the CIR model, the Sharpe ratio for a bond of any maturity is:

$$\phi(r, t) = \bar{\phi} \frac{\sqrt{r}}{\sigma} = 0.04 \frac{\sqrt{0.07}}{0.04} = 0.2646$$

All bonds have the same Sharpe ratio, so the Sharpe ratio of the 9-year bond is 0.2646.

### Solution 36

A Chapter 24, Delta-Gamma Approximation



The process describes the Vasicek model with:

$$a = 0.3, \quad b = 0.1, \quad \sigma = 0.12$$

The short-term interest rate increases by one standard deviation, so under the Vasicek model it increases by:

$$\sigma\sqrt{dt} \approx \sigma\sqrt{h} = 0.12\sqrt{\frac{1}{365}} = 0.00628$$

Itô's Lemma is the basis of the delta-gamma-theta approximation:

$$dP = P_r dr + \frac{1}{2} P_{rr} (dr)^2 + P_t dt$$

Since we are using only the delta-gamma portion of the approximation, we can set the final term to 0. Let's write the delta-gamma approximation in its discrete form:

$$P(0.08628, 0, 10) - P(0.08, 0, 10) \approx (0.00628)P_r + \frac{1}{2}(0.00628)^2 P_{rr}$$

Delta is  $P_r$ , and gamma is  $P_{rr}$ . Both are found below:

$$P(r, t, T) = A(t, T)e^{-B(t, T)r}$$

$$P_r = -B(t, T)P(r, t, T)$$

$$P_{rr} = [B(t, T)]^2 P(r, t, T)$$

$$B(0, 10) = \frac{1 - e^{-0.3(10)}}{0.3} = 3.16738$$

$$P_r(0.08, 0, 10) = -3.16738 \times 0.60024 = -1.90119$$

$$P_{rr}(0.08, 0, 10) = 3.16738^2 \times 0.60024 = 6.02177$$

The delta-gamma approximation of the new price can now be calculated:

$$\begin{aligned} P(0.08628, 0, 10) &= P(0.08, 0, 10) + 0.00628 \times P_r + \frac{1}{2} \times 0.00628^2 P_{rr} \\ &= 0.60024 + 0.00628 \times (-1.90119) + \frac{1}{2} \times 0.00628^2 \times 6.02177 \\ &= 0.588417 \end{aligned}$$

Since the bond matures for \$1,000, its new price is:

$$1,000 \times 0.588417 = 588.417$$

### Solution 37

**A** Chapter 24, Delta-Gamma-Approximation



The process describes the Cox-Ingersoll-Ross model with:

$$a = 0.3, \quad b = 0.1, \quad \sigma = 0.4$$

The Sharpe ratio is zero, so  $\bar{\phi}$  is zero:

$$\phi(r, t) = 0 \quad \Rightarrow \quad \frac{\bar{\phi}\sqrt{r}}{\sigma} = 0 \quad \Rightarrow \quad \bar{\phi} = 0$$

The short-term interest rate increases by one standard deviation, so under the CIR model it increases by:

$$\sigma\sqrt{r \times dt} \approx \sigma\sqrt{r \times h} = 0.4\sqrt{\frac{0.08}{365}} = 0.00592$$

Itô's Lemma is the basis of the delta-gamma-theta approximation:

$$dP = P_r dr + \frac{1}{2} P_{rr} (dr)^2 + P_t dt$$

Since we are using only the delta-gamma portion of the approximation, we can set the final term to 0. Let's write the delta-gamma approximation in its discrete form:

$$P(0.08592, 0, 10) - P(0.08, 0, 10) \approx (0.00592)P_r + \frac{1}{2}(0.00592)^2 P_{rr}$$

We must calculate the value of  $\gamma$  in order to obtain  $B(0, 10)$ :

$$\begin{aligned} \gamma &= \sqrt{(\mathbf{a} - \bar{\phi})^2 + 2\sigma^2} = \sqrt{(0.3 - 0)^2 + 2 \times 0.4^2} = 0.64031 \\ B(0, 10) &= \frac{2(e^{\gamma(T-t)} - 1)}{(\mathbf{a} - \bar{\phi} + \gamma)(e^{\gamma(T-t)} - 1) + 2\gamma} \\ &= \frac{2\left(e^{0.64031 \times (10)} - 1\right)}{(0.3 - 0 + 0.64031)\left(e^{0.64031 \times (10)} - 1\right) + 2(0.64031)} \\ &= \frac{1,205.45657}{568.03352} = 2.12216 \end{aligned}$$

Now we can solve for delta and gamma:

$$\begin{aligned} P(r, t, T) &= A(t, T)e^{-B(t, T)r} \\ P_r &= -B(t, T)P(r, t, T) \\ P_{rr} &= [B(t, T)]^2 P(r, t, T) \\ P_r(0.08, 0, 10) &= -2.12216 \times 0.50045 = -1.06203 \\ P_{rr}(0.08, 0, 10) &= 2.12216^2 \times 0.50045 = 2.25380 \end{aligned}$$

The delta-gamma approximation of the new price can now be calculated:

$$\begin{aligned} P(0.08592, 0, 10) &= P(0.08, 0, 10) + (0.00592)P_r + \frac{1}{2}(0.00592)^2 P_{rr} \\ &= 0.50045 + 0.00592 \times (-1.06203) + \frac{1}{2}(0.00592)^2 \times 2.25380 = 0.494200 \end{aligned}$$

Since the bond matures for \$1000, its new price is:

$$1,000 \times 0.494200 = 494.20$$

**Solution 38****D** Chapter 24, Binomial Interest Rate Model

*This question is similar to Problem 24.7 at the end of the textbook chapter.*

Since interest rates can move up or down by 300 basis points each period, the interest-rate tree is:

|      |      |      |
|------|------|------|
|      |      | 0.17 |
|      | 0.14 |      |
|      |      | 0.11 |
| 0.11 |      |      |
|      |      | 0.11 |
|      | 0.08 |      |
|      |      | 0.05 |

The price of the 3-year zero-coupon bond is:

$$\begin{aligned}
 P_0(0,3) &= E^* \left[ e^{-\sum_{i=0}^2 r_i} \right] \\
 &= 0.25 \times e^{-(0.11+0.14+0.17)} \\
 &\quad + 0.25 \times e^{-(0.11+0.14+0.11)} \\
 &\quad + 0.25 \times e^{-(0.11+0.08+0.11)} \\
 &\quad + 0.25 \times e^{-(0.11+0.08+0.05)} \\
 &= 0.72054
 \end{aligned}$$

The yield of the 3-year zero-coupon bond is:

$$yield = -\frac{\ln[P(0,T)]}{T} = -\frac{\ln(0.72054)}{3} = 0.10925$$

**Solution 39****E** Chapter 24, Black-Derman-Toy Model

The tree of prices for the 3-year bond is:

|        |        |        |
|--------|--------|--------|
|        |        | 0.7717 |
|        | 0.6648 |        |
| 0.6086 |        | 0.8366 |
|        | 0.7351 |        |
|        |        | 0.8858 |

The calculations to obtain these prices are:

$$P(2,3,r_{uu}) = \frac{1}{1.2959} = 0.7717$$

$$P(2,3,r_{ud}) = P(2,3,r_{du}) = \frac{1}{1.1953} = 0.8366$$

$$P(2,3,r_{dd}) = \frac{1}{1.1289} = 0.8858$$

$$P(1,3,r_u) = \frac{0.5(0.7717 + 0.8366)}{1.2097} = 0.6648$$

$$P(1,3,r_d) = \frac{0.5(0.8366 + 0.8858)}{1.1716} = 0.7351$$

$$P(0,3) = \frac{0.5(0.6648 + 0.7351)}{1.15} = 0.6086$$

We don't need to calculate  $P(0,3)$  to answer this question, but we included it for completeness.

The formula for the one-year yield volatility for a  $T$ -year zero-coupon bond is:

$$\text{Yield volatility} = 0.5 \times \ln \left[ \frac{P(1,T,r_u)^{-1/(T-1)} - 1}{P(1,T,r_d)^{-1/(T-1)} - 1} \right]$$

The yield volatility of the 3-year bond is:

$$\text{Yield volatility} = 0.5 \times \ln \left[ \frac{0.6648^{-1/(3-1)} - 1}{0.7351^{-1/(3-1)} - 1} \right] = 0.1542$$

#### Solution 40

A Chapter 24, Black-Derman-Toy Model 

Although the cap payments are made at the end of each year, we will value them at the beginning of each year using the following formula:

$$T\text{-year caplet payoff at time } (T-1) = \frac{\text{Max}[0, R_{T-1} - K_R]}{1 + R_{T-1}} \times \text{Notional}$$

A 4-year cap consists of a 1-year caplet, a 2-year caplet, a 3-year caplet, and a 4-year caplet.

The payoff for the 1-year caplet is zero since 15% is less than 21%.

The payoff for the 2-year caplet is also zero since 20.96% and 17.16% are both less than 21%.

The payoff for the 3-year caplet is positive only when the short-term interest rate increases to 29.59%:

$$R_2 = 0.2959 \Rightarrow \frac{0.2959 - 0.21}{1.2959} \times 100 = 6.6286$$

The payoff for the 4-year caplet is positive only when the short-term rate increases to 25.03%:

$$R_3 = 0.2503 \Rightarrow \frac{0.2503 - 0.21}{1.2503} \times 100 = 3.2232$$

The possible payments from the cap are illustrated in the tree below:

|        |        |        |        |
|--------|--------|--------|--------|
|        |        |        | 3.2232 |
|        |        | 6.6286 |        |
|        | 0.0000 |        | 0.0000 |
| 0.0000 |        | 0.0000 |        |
|        | 0.0000 |        | 0.0000 |
|        |        | 0.0000 |        |
|        |        |        | 0.0000 |

The value of the 3-year caplet is:

$$\text{Value of 3-year caplet} = (0.5)^2 \times \frac{6.6286}{1.15 \times 1.2096} = 1.1913$$

The value of the 4-year caplet is:

$$\text{Value of the 4-year caplet} = (0.5)^3 \times \frac{3.2232}{1.15 \times 1.2096 \times 1.2959} = 0.2235$$

The value of the 4-year cap is:

$$\begin{aligned} & \text{(1-year caplet)} + \text{(2-year caplet)} + \text{(3-year caplet)} + \text{(4-year caplet)} \\ & = 0.000 + 0.0000 + 1.1913 + 0.2235 = 1.4148 \end{aligned}$$

### Solution 41

**C** Chapter 24, Cox-Ingersoll-Ross Model



We begin with the Sharpe ratio and parameterize it for the CIR model:

$$\phi(r, t) = \frac{\alpha(r, t, T) - r}{q(r, t, T)}$$

$$\bar{\phi} \frac{\sqrt{r}}{\sigma} = \frac{\alpha(r, t, T) - r}{B(t, T)\sigma(r)}$$

$$\bar{\phi} r = \frac{\alpha(r, t, T) - r}{B(t, T)}$$

We use the value of  $\alpha(0.08, 0, 3)$  provided in the question:

$$\bar{\phi} \times 0.08 = \frac{0.105324 - 0.08}{B(0, 3)}$$

$$B(0, 3)\bar{\phi} = 0.31655$$

Making use of the fact that  $B(0, 3) = B(2, 5)$ , we have:

$$\bar{\phi} \times 0.09 = \frac{\alpha(0.09, 2, 5) - 0.09}{B(2, 5)}$$

$$\begin{aligned} \alpha(0.09, 2, 5) &= 0.09 + B(2, 5)\bar{\phi}(0.09) \\ &= 0.09 + 0.31655(0.09) \\ &= 0.1184895 \end{aligned}$$

### Solution 42

**D** Chapter 24, Black Formula



The option expires in 1 year, so  $T = 1$ . The underlying bond matures 1 year after the option expires, so  $s = 1$ . The bond forward price is:

$$F = P_0(T, T + s) = \frac{P(0, T + s)}{P(0, T)} = \frac{0.8799}{0.9555} = 0.92088$$

The volatility of the forward price is:

$$\sigma = 0.07$$

We have:

$$d_1 = \frac{\ln\left(\frac{F}{K}\right) + 0.5\sigma^2T}{\sigma\sqrt{T}} = \frac{\ln\left(\frac{0.92088}{0.9111}\right) + 0.5(0.07)^2(1)}{0.07\sqrt{1}} = 0.18752$$

$$d_2 = d_1 - \sigma\sqrt{T} = 0.18752 - 0.07\sqrt{1} = 0.11752$$

$$N(d_1) = N(0.18752) = 0.57437$$

$$N(d_2) = N(0.11752) = 0.54678$$

The Black formula for the call price is:

$$\begin{aligned} C &= P(0, T)[F \times N(d_1) - K \times N(d_2)] \\ &= 0.9555[0.92088 \times 0.57437 - 0.9111 \times 0.54678] \\ &= 0.02939 \end{aligned}$$

**Solution 43****D** Chapter 24, Interest Rate Cap

The tree of interest rates is:

|        |         |         |  |
|--------|---------|---------|--|
|        |         | 13.991% |  |
|        | 12.038% |         |  |
| 7.000% |         | 10.328% |  |
|        | 10.055% |         |  |
|        |         | 7.623%  |  |

Although the cap payments are made at the end of each year, we will follow the textbook's convention of valuing them at the beginning of each year using the following formula:

$$T\text{-year caplet payoff at time } (T - 1) = \frac{\text{Max}[0, R_{T-1} - K_R]}{1 + R_{T-1}} \times \text{Notional}$$

The interest rate cap consists of a 1-year caplet, a 2-year caplet, and a 3-year caplet.

The payoff for the 1-year caplet is zero since 7% is less than 11.0%.

The payoff for the 2-year caplet is positive only when the short-term interest rate increases to 12.038%, since 10.055% is less than 11%:

$$R_1 = 0.12038 \Rightarrow \frac{0.12038 - 0.11}{1.12038} \times 100 = 0.9265$$

The payoff for the 3-year caplet is positive only when the short-term rate increases to 13.991%, since the other short-term interest rates at that time are less than 11%:

$$R_2 = 0.13991 \Rightarrow \frac{0.13991 - 0.11}{1.13991} \times 100 = 2.6239$$

The payoff table is illustrated in the tree below:

|        |        |        |  |
|--------|--------|--------|--|
|        |        | 2.6239 |  |
|        | 0.9265 |        |  |
| 0.0000 |        | 0.0000 |  |
|        | 0.0000 |        |  |
|        |        | 0.0000 |  |

The value of the 2-year caplet is:

$$\text{Value of 2-year caplet} = 0.5 \times \frac{0.9265}{1.07} = 0.4329$$

The value of the 3-year caplet is:

$$\text{Value of the 3-year caplet} = (0.5)^2 \times \frac{2.6239}{1.07 \times 1.12038} = 0.5472$$

The value of the 3-year cap is:

$$\begin{aligned} & (1\text{-year caplet}) + (2\text{-year caplet}) + (3\text{-year caplet}) \\ & = 0.0000 + 0.4329 + 0.5472 = 0.9801 \end{aligned}$$

#### Solution 44

**B** Chapter 24, Vasicek Model



In the Vasicek Model, we have:

$$P(r, t, T) = A(t, T)e^{-B(t, T)r}$$

We use the following two facts about the Vasicek model:

- $A(t, T)$  and  $B(t, T)$  do not depend on  $r$ .
- $A(t, T) = A(0, T - t)$  and  $B(t, T) = B(0, T - t)$ . This implies:

$$\begin{aligned} A(0, 3) &= A(2, 5) = A(3, 6) \\ B(0, 3) &= B(2, 5) = B(3, 6) \end{aligned}$$

Since  $A(0, 3) = A(2, 5)$ , we have two equations and two unknowns:

$$\begin{aligned} A(0, 3)e^{-B(0, 3)(0.08)} &= 0.8782 \\ A(0, 3)e^{-B(0, 3)(0.10)} &= 0.8395 \end{aligned}$$

Dividing the second equation into the first equation allows us to find  $B(0, 3)$ :


$$\begin{aligned} e^{-B(0, 3)(0.08) + B(0, 3)(0.10)} &= \frac{0.8782}{0.8395} \\ B(0, 3)(0.02) &= \ln\left(\frac{0.8782}{0.8395}\right) \\ B(0, 3) &= 2.25339 \end{aligned}$$

We can now solve for the value of  $A(0, 3)$ :

$$\begin{aligned} A(0, 3)e^{-B(0, 3)(0.08)} &= 0.8782 \\ A(0, 3) &= 0.8782e^{B(0, 3)(0.08)} \\ A(0, 3) &= 0.8782e^{2.25339(0.08)} \\ A(0, 3) &= 1.05168 \end{aligned}$$

We can now solve for  $r^*$ :

$$\begin{aligned} A(0, 3)e^{-B(0, 3)(r^*)} &= 0.8586 \\ 1.05168e^{-2.25339r^*} &= 0.8586 \\ -2.25339r^* &= \ln\left(\frac{0.8586}{1.05168}\right) \\ r^* &= 0.0900 \end{aligned}$$

**Solution 45****B** Chapter 24, Vasicek Model 

The process describes the Vasicek model with:

$$a = 0.2$$

$$b = 0.1$$

$$\sigma = 0.02$$

The Sharpe ratio is:

$$\phi = 0.05$$

We can use these values to find the price of the 10-year zero-coupon bond:

$$\bar{r} = b + \phi \frac{\sigma}{a} - 0.5 \frac{\sigma^2}{a^2} = 0.1 + 0.05 \left( \frac{0.02}{0.2} \right) - 0.5 \frac{0.02^2}{0.2^2} = 0.1000$$


$$B(t, T) = \frac{1 - e^{-a(T-t)}}{a} = \frac{1 - e^{-0.2(10)}}{0.2} = 4.32332$$

$$A(t, T) = e^{\bar{r} \times [B(t, T) - (T-t)] - \frac{[B(t, T)]^2 \times \sigma^2}{4a}} = e^{0.1000(4.32332 - 10) - \frac{4.32332^2 \times 0.02^2}{4 \times 0.2}} = 0.56157$$

$$P(r, t, T) = A(t, T) e^{-B(t, T)r} = 0.56157 e^{-4.32332 \times 0.08} = 0.39737$$

Since the bond matures for \$100, its price is:

$$100 \times 0.39737 = 39.737$$

**Solution 46****C** Chapter 24, Vasicek Model 

The process describes the Vasicek model with:

$$a = 0.15$$

$$b = 0.12$$

$$\sigma = 0.03$$

$$\phi = 0.07$$

Since the current value of the short-term interest rate is 8%:

$$a(r) = a \times (b - r) = 0.15 \times (0.12 - 0.08) = 0.006$$

$$\sigma(r) = \sigma = 0.03$$

We can use the following relationship to find the value of theta,  $P_t$ :

$$rP = \frac{1}{2} [\sigma(r)]^2 P_{rr} + [a(r) + \sigma(r)\phi(r, t)] P_r + P_t$$

We can use the formula for the price of a zero-coupon bond to find the first and second derivative with respect to the short-term interest rate:

$$\begin{aligned} P(r, t, T) &= A(t, T)e^{-B(t, T)r} \\ P_r &= -B(t, T)A(t, T)e^{-B(t, T)r} = -B(t, T)P(r, t, T) \\ P_{rr} &= [B(t, T)]^2 A(t, T)e^{-B(t, T)r} = [B(t, T)]^2 P(r, t, T) \end{aligned}$$

The value of  $B(0, 10)$  is:

$$B(0, 10) = \frac{1 - e^{-a(T-t)}}{a} = \frac{1 - e^{-0.15(10-0)}}{0.15} = 5.17913$$

We have:

$$\begin{aligned} P(0.08, 0, 10) &= 0.36636 \\ P_r &= -B(t, T)P(r, t, T) = -5.17913 \times 0.36636 = -1.89743 \\ P_{rr} &= [B(t, T)]^2 P(r, t, T) = 5.17913^2 \times 0.36636 = 9.82702 \end{aligned}$$

We can now find theta:

$$\begin{aligned} rP &= \frac{1}{2} [\sigma(r)]^2 P_{rr} + [a(r) + \sigma(r)\phi(r, t)]P_r + P_t \\ 0.08(0.36636) &= \frac{1}{2} [0.03]^2 (9.82702) + [0.006 + (0.03)(0.07)](-1.89743) + P_t \\ P_t &= 0.0403 \end{aligned}$$

### Solution 47

**A** Chapter 24, Risk-Neutral Interest Rate Models



Zachary is using the Vasicek model:

$$\begin{aligned} dr &= a(b - r)dt + \sigma dZ(t) \\ dr &= 0 \times (0.10 - r)dt + 0.05dZ(t) \\ dr &= 0.05dZ(t) \end{aligned}$$

Although Zachary's model resembles Charlie's model, Zachary's model uses a non-zero Sharpe ratio:

$$\phi = 0.20$$

Therefore, Zachary's model is not risk-neutral.

Charlie's model is a risk-neutral version of the Vasicek model, so Charlie uses a Sharpe ratio of zero:

$$\phi = 0.00$$

Therefore, Charlie's model is different from Zachary's model.

The risk-neutral version of Zachary's model is:

$$\begin{aligned} dr &= [a(r) + \sigma(r)\phi(r,t)]dt + \sigma(r)d\tilde{Z}(t) \\ &= [a(b-r) + \sigma\phi]dt + \sigma d\tilde{Z}(t) \\ &= [0 \times (0.10 - r) + 0.05 \times 0.20]dt + 0.05d\tilde{Z}(t) \\ &= 0.01dt + 0.05d\tilde{Z}(t) \end{aligned}$$

This risk-neutral process is the same as the risk-neutral process used by Ann. Therefore, Choice (A) is the correct answer.

### Solution 48

A Chapter 24, Delta-Gamma-Theta Approximation



The interest rate process describes the Vasicek model. In this model, we have:

$$\begin{aligned} P(r,t,T) &= A(t,T)e^{-B(t,T)r} \\ P_r(r,t,T) &= -B(t,T)A(t,T)e^{-B(t,T)r} \\ P_{rr}(r,t,T) &= [B(t,T)]^2 A(t,T)e^{-B(t,T)r} \end{aligned}$$

We can calculate the value of  $B(t,T)$ :

$$B(0,5) = \frac{1 - e^{-a(T-t)}}{a} = \frac{1 - e^{-0.25(5-0)}}{0.25} = 2.85398$$

We are given  $P_t$  in the question, and we can calculate  $P_r$  and  $P_{rr}$ :

$$\begin{aligned} P_r(0.07, 0, 5) &= -B(0,5)A(0,5)e^{-B(0,5)r} = -2.85398 \times 0.5961 = -1.70126 \\ P_{rr}(0.07, 0, 5) &= [B(0,5)]^2 A(0,5)e^{-B(0,5)r} = 2.85398^2 \times 0.5961 = 4.85536 \\ P_t(0.07, 0, 5) &= 0.0685 \end{aligned}$$

A one standard deviation movement in the short-term interest rate over one day is:

$$\sigma\sqrt{dt} \approx \sigma\sqrt{h} = 0.10 \times \sqrt{\frac{1}{365}} = 0.0052342$$


We can use the delta-gamma-theta approximation to estimate the resulting change in the price:

$$\begin{aligned} dP &= P_r dr + \frac{1}{2} P_{rr} (dr)^2 + P_t dt \\ &\approx -1.70126(0.0052342) + \frac{1}{2}(4.85536)(0.0052342)^2 + 0.0685 \left( \frac{1}{365} \right) = -0.008651 \end{aligned}$$

The new price is:

$$0.5961 - 0.008651 = 0.5874$$

**Solution 49**

**C** Chapter 24, Black Model 

We need the volatility of a forward that matures at time 2 and has a bond maturing at time 4 as its underlying asset:

$$\sigma = \sqrt{\frac{\text{Var}\{\ln[P_t(T, T+s)]\}}{t}} = \sqrt{\frac{\text{Var}\{\ln[P_t(2, 4)]\}}{t}} = \sqrt{\frac{0.0289t}{t}} = 0.17$$

The bond forward price is:

$$F = P_0(T, T+s) = \frac{P(0, T+s)}{P(0, T)} = \frac{P(0, 4)}{P(0, 2)} = \frac{0.6956}{0.8573} = 0.81138$$

The values of  $d_1$  and  $d_2$  are:

$$d_1 = \frac{\ln\left(\frac{F}{K}\right) + 0.5\sigma^2 T}{\sigma\sqrt{T}} = \frac{\ln\left(\frac{0.81138}{0.92}\right) + 0.5(0.17)^2(2)}{0.17\sqrt{2}} = -0.40235$$

$$d_2 = d_1 - \sigma\sqrt{T} = -0.40235 - 0.17\sqrt{2} = -0.64277$$

We have:

$$N(d_1) = N(-0.40235) = 0.34371$$

$$N(d_2) = N(-0.64277) = 0.26019$$

The Black formula for the call price is:

$$C = P(0, T)[F \times N(d_1) - K \times N(d_2)]$$

$$= 0.8573[0.81138 \times 0.34371 - 0.92 \times 0.26019]$$

$$= 0.03387$$

**Solution 50**

**E** Chapter 24, Cox-Ingersoll-Ross Model 

We begin with the Sharpe ratio and parameterize it for the Cox-Ingersoll-Ross model:

$$\phi(r, t) = \frac{\alpha(r, t, T) - r}{q(r, t, T)}$$

$$\bar{\phi} \frac{\sqrt{r}}{\sigma} = \frac{\alpha(r, t, T) - r}{B(t, T)\sigma(r)}$$

$$\bar{\phi} r = \frac{\alpha(r, t, T) - r}{B(t, T)}$$

We can substitute 7.0% for  $\alpha(0.05,0,10)$ :

$$\begin{aligned}\bar{\phi} \times 0.05 &= \frac{0.07 - 0.05}{B(0,10)} \\ 0.07 &= 0.05 + B(0,10)\bar{\phi}(0.05) \\ B(0,10)\bar{\phi} &= 0.40\end{aligned}$$

Making use of the fact that  $B(0,10) = B(5,15)$ , we have:

$$\begin{aligned}\bar{\phi} \times 0.06 &= \frac{\alpha(0.06,5,15) - 0.06}{B(5,15)} \\ \alpha(0.06,5,15) &= 0.06 + B(5,15)\bar{\phi}(0.06) \\ &= 0.06 + 0.40(0.06) \\ &= 0.084\end{aligned}$$

### Solution 51

**C** Chapter 24, Vasicek Model



The Vasicek model of short-term interest rates is:

$$dr = a(b - r)dt + \sigma dZ$$

Therefore, we can determine the value of  $a$ :

$$dr = 0.35(b - r)dt + \sigma dZ \quad \Rightarrow \quad a = 0.35$$

In the Vasicek model, the Sharpe ratio is constant:

$$\phi(r, t) = \phi$$

Therefore, for any  $r$ ,  $t$ , and  $T$ , we have:

$$\phi = \frac{\alpha(r, t, T) - r}{q(r, t, T)}$$

Since the Sharpe ratio is constant:

$$\frac{\alpha(0.06, 0, 5) - 0.06}{q(0.06, 0, 5)} = \frac{\alpha(0.03, 2, 8) - 0.03}{q(0.03, 2, 8)}$$

We now make use of the following formula for  $q(r, t, T)$ :

$$q(r, t, T) = B(t, T)\sigma(r) = B(t, T)\sigma$$

Substituting this expression for  $q(r,t,T)$  into the preceding equation allows us to solve for  $\alpha(0.03,2,8)$ :

$$\begin{aligned} \frac{\alpha(0.06,0,5) - 0.06}{q(0.06,0,5)} &= \frac{\alpha(0.03,2,8) - 0.03}{q(0.03,2,8)} \\ \frac{\alpha(0.06,0,5) - 0.06}{B(0,5)\sigma} &= \frac{\alpha(0.03,2,8) - 0.03}{B(2,8)\sigma} \\ \frac{0.073 - 0.06}{1 - e^{-0.35(5-0)}} &= \frac{\alpha(0.03,2,8) - 0.03}{1 - e^{-0.35(8-2)}} \\ \frac{0.013}{0.35} &= \frac{\alpha(0.03,2,8) - 0.03}{0.35} \\ \frac{0.013}{2.3606} &= \frac{\alpha(0.03,2,8) - 0.03}{2.5073} \\ \alpha(0.03,2,8) &= 0.0438 \end{aligned}$$

### Solution 52

**D** Chapter 24, Interest Rate Derivative



The realistic process for the short rate follows:

$$dr = a(r)dt + \sigma(r)dZ \quad \text{where:} \quad a(r) = 0.04 - 0.25r \quad \& \quad \sigma(r) = 0.4$$

The risk-neutral process follows:

$$dr = [a(r) + \sigma(r)\phi(r,t)]dt + \sigma(r)d\tilde{Z}$$

We can use the coefficient of the first term of the risk-neutral process to solve for the Sharpe ratio,  $\phi(r,t)$ :


$$\begin{aligned} 0.10 - 0.25r &= a(r) + \sigma(r)\phi(r,t) \\ 0.10 - 0.25r &= 0.04 - 0.25r + 0.4\phi(r,t) \\ \phi(r,t) &= 0.15 \end{aligned}$$

The derivative, like all interest-rate dependent assets, must have a Sharpe ratio of 0.15. Let's rearrange the differential equation for  $g$ , so that we can more easily observe its Sharpe ratio:

$$\frac{dg}{g} = (r + 0.09)dt - \beta dZ \quad \Rightarrow \quad \phi(r,t) = \frac{(r + 0.09) - r}{\beta} = \frac{0.09}{\beta}$$

Since the Sharpe ratio is 0.15:

$$\begin{aligned} 0.15 &= \frac{0.09}{\beta} \\ \beta &= 0.6 \end{aligned}$$

**Solution 53****E** Chapter 24, Interest Rate Derivative 

Let's rearrange the differential equation for  $g$ , so that we can more easily observe its Sharpe ratio:

$$\frac{dg}{g} = (r + 0.02)dt - 0.08dZ \Rightarrow \phi(r, t) = \frac{(r + 0.02) - r}{0.08} = 0.25$$

The realistic process for the short rate follows:

$$dr = a(r)dt + \sigma(r)dZ \quad \text{where:} \quad a(r) = 0.04 - 0.25r \quad \& \quad \sigma(r) = 0.4$$

The risk-neutral process follows:

$$dr = [a(r) + \sigma(r)\phi(r, t)]dt + \sigma(r)d\tilde{Z}$$

We can now find the coefficient of the first term of the risk-neutral process:

$$\begin{aligned} \mu(r) &= a(r) + \sigma(r)\phi(r, t) \\ &= 0.04 - 0.25r + 0.4 \times 0.25 \\ &= 0.14 - 0.25r \end{aligned}$$

**Solution 54****E** Chapter 24, Black-Derman-Toy Model 

Each node is  $e^{2\sigma_i\sqrt{h}}$  times the one below it. This means that 38.75% is  $e^{4\sigma_2}$  times as large as 10.08%:

$$\begin{aligned} 0.3875 &= 0.1008e^{4\sigma_2} \\ \sigma_2 &= \ln\left(\frac{0.3875}{0.1008}\right) \times \frac{1}{4} = 0.33664 \end{aligned}$$

The interest rate for the node above 10.08% is:

$$0.1008e^{2\sigma_2} = 0.1008e^{2 \times 0.33664} = 0.1976$$

The tree of prices for the 3-year bond is:

|   |        |        |
|---|--------|--------|
|   |        | 0.7207 |
|   | 0.6567 | 0.8350 |
| ? | 0.7669 | 0.9084 |

The calculations to obtain these prices are:

$$P(2,3,r_{uu}) = \frac{1}{1.3875} = 0.7207$$

$$P(2,3,r_{ud}) = P(2,3,r_{du}) = \frac{1}{1.1976} = 0.8350$$

$$P(2,3,r_{dd}) = \frac{1}{1.1008} = 0.9084$$

$$P(1,3,r_u) = \frac{0.5(0.7207 + 0.8350)}{1.1845} = 0.6567$$

$$P(1,3,r_d) = \frac{0.5(0.8350 + 0.9084)}{1.1367} = 0.7669$$

We cannot calculate the current price of the bond,  $P(0,3)$  because we do not know the value of  $r_0$ , but we do not need  $P(0,3)$  to answer this question.

The formula for the one-year yield volatility for a  $T$ -year zero-coupon bond is:

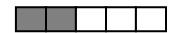
$$\text{Yield volatility} = 0.5 \times \ln \left[ \frac{P(1,T,r_u)^{-1/(T-1)} - 1}{P(1,T,r_d)^{-1/(T-1)} - 1} \right]$$

The yield volatility of the 3-year bond is:

$$\text{Yield volatility} = 0.5 \times \ln \left[ \frac{0.6567^{-1/(3-1)} - 1}{0.7669^{-1/(3-1)} - 1} \right] = 0.2501$$

### Solution 55

**D** Chapter 24, Black-Derman-Toy Model



The value of  $r_0$  is the same as the yield on the 1-year zero-coupon bond, and the yield volatility of the 2-year bond is  $\sigma_1$ :

$$\frac{1}{1+r_0} = 0.9346 \quad \Rightarrow \quad r_0 = 7\%$$

$$\sigma_1 = 9\%$$

The first part of the interest rate tree is:

$$\begin{array}{ccc} r_u = R_1 e^{2\sigma_1} & & r_d e^{0.18} \\ r_0 & \Rightarrow & 7\% \\ r_d = R_1 & & r_d \end{array}$$

We need to determine the value of  $r_d$ . The tree must correctly price the 2-year zero-coupon bond, so:

$$P(0,2) = \frac{1}{1.07}(0.5) \left( \frac{1}{1+r_d e^{0.18}} + \frac{1}{1+r_d} \right)$$

$$0.8264 = 0.9346(0.5) \left( \frac{1}{1+r_d e^{0.18}} + \frac{1}{1+r_d} \right)$$

The quickest way to answer this question is to use trial and error.

Let's try Choice C first:

$$0.9346(0.5) \left( \frac{1}{1+(0.10)e^{0.18}} + \frac{1}{1+0.10} \right) = 0.8422$$

Since the 0.8422 is greater than 0.8264, let's try a higher short rate.

Let's try Choice D:

$$0.9346(0.5) \left( \frac{1}{1+(0.1193)e^{0.18}} + \frac{1}{1+0.1193} \right) = 0.8264$$

Choice D is the correct answer.

### Solution 56

**E** Chapters 20 and 24, CIR Model



Let's find the differential form of each choice.

Choice A does not contain a random variable, so its differential form is easy to find:

$$dr(t) = \left[ -abr(0)e^{-at} + a\sigma e^{-at} \right] dt$$

Choice B is easily recognized as an arithmetic Brownian motion:

$$dr(t) = bdt + \sigma dZ(t)$$

Choice C is easily recognized as a geometric Brownian motion:

$$dr(t) = br(t)dt + \sigma r(t)dZ(t)$$

Choice D is (perhaps not so easily) recognized as an Ornstein-Uhlenbeck process:

$$dr(t) = a \times [b - r(t)]dt + \sigma dZ(t)$$

Choice D therefore describes the Vasicek Model.

By the process of elimination, we see that Choice E must be the correct answer. But for the sake of thoroughness, let's find the differential of Choice E as well.

The first two terms of Choice E do not contain random variables, so their differentials are easy to find. The third term will be more difficult:

$$r(t) = r(0)e^{-at} + b(1 - e^{-at}) + \sigma \int_0^t e^{a(s-t)} \sqrt{r(s)} dZ(s)$$

$$dr(t) = -ar(0)e^{-at} dt + abe^{-at} dt + d \left[ \sigma \int_0^t e^{a(s-t)} \sqrt{r(s)} dZ(s) \right]$$

The third term has a function of  $t$  in the integral. We can pull the  $t$ -dependent portion out of the integral, so that we are finding the differential of a product. We then use the following version of the product rule to find the differential:

$$d[U(t)V(t)] = dU(t)V(t) + U(t)dV(t)$$

The differential of the third term is:

$$d \left[ \sigma \int_0^t e^{a(s-t)} \sqrt{r(s)} dZ(s) \right] = d \left[ \sigma e^{-at} \int_0^t e^{as} \sqrt{r(s)} dZ(s) \right]$$

$$= -a\sigma e^{-at} dt \int_0^t e^{as} \sqrt{r(s)} dZ(s) + \sigma e^{-at} e^{at} \sqrt{r(t)} dZ(t)$$

$$= -a\sigma e^{-at} dt \int_0^t e^{as} \sqrt{r(s)} dZ(s) + \sigma \sqrt{r(t)} dZ(t)$$

Putting all three terms together, we have:

$$dr(t) = -ar(0)e^{-at} dt + abe^{-at} dt - a\sigma e^{-at} dt \int_0^t e^{as} \sqrt{r(s)} dZ(s) + \sigma \sqrt{r(t)} dZ(t)$$

$$= -a \left[ r(0)e^{-at} - be^{-at} + \sigma e^{-at} \int_0^t e^{as} \sqrt{r(s)} dZ(s) \right] dt + \sigma \sqrt{r(t)} dZ(t)$$

$$= -a \left[ r(0)e^{-at} + b - b - be^{-at} + \sigma e^{-at} \int_0^t e^{as} \sqrt{r(s)} dZ(s) \right] dt + \sigma \sqrt{r(t)} dZ(t)$$

$$= -a \left[ r(0)e^{-at} + b - be^{-at} + \sigma e^{-at} \int_0^t e^{as} \sqrt{r(s)} dZ(s) \right] dt + abdt + \sigma \sqrt{r(t)} dZ(t)$$

$$= -a \left[ r(0)e^{-at} + b(1 - e^{-at}) + \sigma \int_0^t e^{a(s-t)} \sqrt{r(s)} dZ(s) \right] dt + abdt + \sigma \sqrt{r(t)} dZ(t)$$

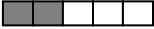
The portion in the brackets above is equal to the expression for  $r(t)$  provided in the Choice E:

$$dr(t) = -ar(t)dt + abdt + \sigma \sqrt{r(t)} dZ(t)$$

$$= abdt - ar(t)dt + \sigma \sqrt{r(t)} dZ(t)$$

$$= a[b - r(t)]dt + \sigma \sqrt{r(t)} dZ(t)$$

**Solution 57**

**C** Chapter 24, Delta-Gamma Approximation for Bonds 

In the model described in the question, we have:

$$P(r, t, T) = A(t, T)e^{-B(t, T)r}$$

$$P_r(r, t, T) = -B(t, T)A(t, T)e^{-B(t, T)r} = -B(t, T)P(r, t, T)$$

$$P_{rr}(r, t, T) = [B(t, T)]^2 A(t, T)e^{-B(t, T)r} = [B(t, T)]^2 P(r, t, T)$$

The delta-gamma-theta approximation is:

$$P[r(t+h), t+h, T] - P[r(t), t, T] \approx [r(t+h) - r(t)]P_r + 0.5[r(t+h) - r(t)]^2 P_{rr} + P_t h$$

This question asks us to use only the delta and gamma portions of the approximation, so we make the following adjustments to the formula above:

- The theta term,  $P_t h$ , is removed from the expression above.
- We replace  $r(t)$  with 0.06.
- We replace  $r(t+h)$  with 0.02.

The delta-gamma approximation is therefore:

$$P_{Est}(0.02, 0, 2) - P(0.06, 0, 2) \approx [-0.04]P_r(0.06, 0, 2) + 0.5[-0.04]^2 P_{rr}(0.06, 0, 2)$$

$$P_{Est}(0.02, 0, 2) \approx P(0.06, 0, 2) + [-0.04]P_r(0.06, 0, 2) + 0.5[-0.04]^2 P_{rr}(0.06, 0, 2)$$

$$P_{Est}(0.02, 0, 2) \approx P(0.06, 0, 2)$$

$$-[-0.04]B(0, 2)P(0.06, 0, 2) + 0.5[-0.04]^2 [B(0, 2)]^2 P(0.06, 0, 2)$$


$$P_{Est}(0.02, 0, 2) \approx P(0.06, 0, 2) \left[ 1 + 0.04 \times 3 + 0.5 \times (-0.04)^2 \times 3^2 \right]$$

$$P_{Est}(0.02, 0, 2) \approx P(0.06, 0, 2) \times 1.1272$$

Therefore, we have:

$$\frac{P_{Est}(0.02, 0, 2)}{P(0.06, 0, 2)} \approx \frac{P(0.06, 0, 2) \times 1.1272}{P(0.06, 0, 2)} = 1.1272$$

**Solution 58**

**A** Chapters 10 & 24, Risk-Neutral Probability 

Let's use  $P(0, 2)$  to denote the price of a 2-year zero-coupon bond that matures for \$1.

We can make use of put-call parity:

$$C(69) + 69 \times P(0, 2) = S_0 + P(69)$$

$$P(69) - C(69) = 69 \times P(0, 2) - 75$$

We can use the stock prices to determine the risk-neutral probability that the up state of the world occurs:

$$p^* = \frac{e^{(r-\delta)h} - d}{u - d} = \frac{(1+r_0)^h - d}{u - d} = \frac{(1.06)^1 - 50/75}{100/75 - 50/75} = \frac{0.3933}{0.6667} = 0.59$$

If the up state occurs, then the zero-coupon bond will have a value of  $\frac{1}{1.075}$  at time 1, and

if the down state occurs, then the zero-coupon bond will have a value of  $\frac{1}{1.045}$  at time 1.

The time 0 value is found using the risk-neutral probabilities and the risk-free rate at time 0:

$$P(0,2) = \frac{1}{1.06} \left[ 0.59 \times \frac{1}{1.075} + 0.41 \times \frac{1}{1.045} \right] = 0.88791$$

We can now use the equation for put-call parity described above to find the solution:

$$P(69) - C(69) = 69 \times P(0,2) - 75 = 69 \times 0.88791 - 75 = -13.7344$$

### Solution 59

**D** Chapter 24, Theta in the Cox-Ingersoll-Ross Model



The process describes the Cox-Ingersoll-Ross-Model with:

$$a(r) = a \times (b - r) = 0.09(0.11 - r)$$

$$\sigma(r) = \sigma\sqrt{r} = 0.22\sqrt{r}$$

The price of a 5-year bond is:

$$P(r,t,T) = A(t,T)e^{-B(t,T)r} = 0.9054e^{-3.4854 \times 0.06} = 0.7345$$

The formula for the price of a bond must satisfy the following partial differential equation:

$$rP = \frac{1}{2}[\sigma(r)]^2 P_{rr} + [a(r) + \sigma(r)\phi(r,t)]P_r + P_t$$

Delta and gamma for the bond are:

$$P(r,t,T) = A(t,T)e^{-B(t,T)r}$$

$$\text{Delta: } P_r = -B(t,T)P(r,t,T) = -3.4854 \times 0.7345 = -2.5602$$

$$\text{Gamma: } P_{rr} = [B(t,T)]^2 P(r,t,T) = (3.4854)^2 \times 0.7345 = 8.9233$$

We can now solve for theta:

$$rP = \frac{1}{2}[\sigma(r)]^2 P_{rr} + [a(r) + \sigma(r)\phi(r,t)]P_r + P_t$$

$$0.06 \times 0.7345$$

$$= \frac{1}{2}(0.22\sqrt{0.06})^2 \times 8.9233 + [0.09(0.11 - 0.06) + 0.22\sqrt{0.06} \times (0.00)](-2.5602) + P_t$$

$$P_t = 0.0426$$

### Solution 60

**B** Chapter 24, Caplet in Black-Derman-Toy Model



In each column of rates, each rate is greater than the rate below it by a factor of:

$$e^{2\sigma_i\sqrt{h}}$$

Therefore, the missing rate in the third column is:

$$0.2290e^{-2\sigma_i\sqrt{1}} = 0.2290 \times \frac{0.2290}{0.3001} = 0.1747$$

The missing rate in the fourth column is:

$$0.3582e^{-2\sigma_i\sqrt{1}} = 0.3582 \times \frac{0.1451}{0.1962} = 0.2649$$

The tree of short-term rates is:

|        |        |        |        |
|--------|--------|--------|--------|
|        |        |        | 35.82% |
|        |        | 30.01% |        |
|        | 21.16% |        | 26.49% |
| 15.00% |        | 22.90% |        |
|        | 16.98% |        | 19.62% |
|        |        | 17.47% |        |
|        |        |        | 14.51% |

The caplet pays off only if the interest rate at the end of the third year is greater than 20%. The payoff table is:

| <u>Time 0</u> | <u>Time 1</u> | <u>Time 2</u> | <u>Time 3</u> |
|---------------|---------------|---------------|---------------|
|               |               |               | 11.6478       |
|               |               | 0.0000        |               |
|               | 0.0000        |               | 5.1314        |
| 0.0000        |               | 0.0000        |               |
|               | 0.0000        |               | 0.0000        |
|               |               | 0.0000        |               |
|               |               |               | 0.0000        |

The payments have been converted to their equivalents payable at the end of 3 years. The calculations for the tree above are shown below:

$$\frac{100 \times (0.3582 - 0.20)}{1.3582} = 11.6478$$

$$\frac{100 \times (0.2649 - 0.20)}{1.2649} = 5.1314$$

We work recursively to calculate the value of the caplet. For example, if the interest rate increases at the end of the first year and at the end of the second year, then its value at the end of the second year is:

$$\frac{0.5 \times 11.6478 + 0.5 \times 5.1314}{1.3001} = 6.4530$$

The completed tree is:

| <u>Time 0</u> | <u>Time 1</u> | <u>Time 2</u> | <u>Time 3</u> |
|---------------|---------------|---------------|---------------|
|               |               |               | 11.6478       |
|               |               | 6.4530        |               |
|               | 3.5245        |               | 5.1314        |
| 1.9204        |               | 2.0876        |               |
|               | 0.8923        |               | 0.0000        |
|               |               | 0.0000        |               |
|               |               |               | 0.0000        |

The value of the year-4 caplet is \$1.9204.

### Solution 61

C Chapter 24, Cox-Ingersoll-Ross Model



The CIR model is:

$$dr = a(b - r)dt + \sigma\sqrt{r}dZ$$

From the partial differential equation provided in the question, we can obtain the following parameters for the CIR model:

$$a = 0.12 \qquad b = \frac{0.0096}{0.12} = 0.08 \qquad \sigma = 0.10$$

We can define  $c$  in terms of  $\bar{\phi}$ :

$$\phi(r, t) = \bar{\phi} \frac{\sqrt{r}}{\sigma} \quad \& \quad \phi(r, t) = c\sqrt{r} \quad \Rightarrow \quad c = \frac{\bar{\phi}}{\sigma}$$

In the CIR model, as the maturity of a zero-coupon bond approaches infinity, its yield approaches:

$$\bar{r} = \frac{2ab}{(a - \bar{\phi} + \gamma)} = \lim_{T \rightarrow \infty} \left[ \frac{-\ln[P(r, t, T)]}{T - t} \right]$$

From the information provided in part (ii) of the question, we know that  $\bar{r}$  is equal to 0.0725. Therefore:

$$\begin{aligned}\frac{2ab}{(a - \bar{\phi} + \gamma)} &= 0.0725 \\ \frac{2 \times 0.12 \times 0.08}{(0.12 - \bar{\phi} + \gamma)} &= 0.0725 \\ 0.0192 &= 0.0725(0.12 - \bar{\phi} + \gamma) \\ 0.26483 &= 0.12 - \bar{\phi} + \gamma \\ \gamma &= \bar{\phi} + 0.14483\end{aligned}$$

We can substitute this value into the formula for  $\gamma$ :

$$\begin{aligned}\gamma &= \sqrt{(a - \bar{\phi})^2 + 2\sigma^2} \\ \bar{\phi} + 0.14483 &= \sqrt{(0.12 - \bar{\phi})^2 + 2(0.1)^2} \\ \bar{\phi}^2 + 0.28966\bar{\phi} + 0.020975 &= (0.12 - \bar{\phi})^2 + 2(0.10)^2 \\ \bar{\phi}^2 + 0.28966\bar{\phi} + 0.020975 &= (0.0144 - 0.24\bar{\phi} + \bar{\phi}^2) + 0.02 \\ 0.52966\bar{\phi} &= 0.013425 \\ \bar{\phi} &= 0.025347\end{aligned}$$

The value of  $c$  is:

$$c = \frac{\bar{\phi}}{\sigma} = \frac{0.025347}{0.10} = 0.25347$$

### Solution 62

**A** Chapter 24, Vasicek Model



The Vasicek model is:

$$dr = a(b - r)dt + \sigma dZ$$

From the partial differential equation provided in the question, we can obtain the following parameters for the CIR model:

$$a = 0.30 \quad b = \frac{0.03}{0.3} = 0.10 \quad \sigma = 0.12$$

In the Vasicek model, as the maturity of a zero-coupon bond approaches infinity, its yield approaches:

$$\bar{r} = b + \phi \frac{\sigma}{a} - 0.5 \frac{\sigma^2}{a^2} = \text{Lim}_{T \rightarrow \infty} \left[ \frac{-\ln[P(r, t, T)]}{T - t} \right]$$

From the information provided the question, we know that  $\bar{r}$  is equal to 0.04. Therefore:

$$b + \phi \frac{\sigma}{a} - 0.5 \frac{\sigma^2}{a^2} = 0.04$$

$$0.1 + \phi \frac{0.12}{0.30} - 0.5 \frac{0.12^2}{0.30^2} = 0.04$$

$$\phi = 0.05$$

### Solution 63

**D** Chapter 24, Binomial Interest Rate Model



A 1-year bond in 2 years is a 3-year bond now. The tree of prices for the 3-year bond is:

|        |        |        |
|--------|--------|--------|
|        | 0.8353 | 1.0000 |
| 0.7278 |        |        |
|        | 0.8869 | 1.0000 |
| 0.6658 |        |        |
|        | 0.9048 | 1.0000 |
| 0.8421 |        |        |
|        | 0.9418 | 1.0000 |

*There is no need to calculate the first two columns of the tree to answer this question, but they are shown above for illustrative purposes.*

The third column is easily obtained by discounting a 1-year bond at the possible interest rates at time 2:

$$e^{-0.18} = 0.8353$$

$$e^{-0.12} = 0.8869$$

$$e^{-0.10} = 0.9048$$

$$e^{-0.06} = 0.9418$$

The put option pays off only if the time-2 price falls below \$0.90. This occurs at 2 nodes, and the payoffs are:

$$\text{up-up: } V_2 = 0.90 - 0.8353 = 0.0647$$

$$\text{up-down: } V_2 = 0.90 - 0.8869 = 0.0131$$

The price of the put option is:

$$V_0 = E^* \left[ V_2 \times e^{-\sum_{i=0}^{2-1} r_i} \right] = \frac{(0.8)^2 0.0647}{e^{(0.12+0.15)}} + \frac{(0.8)(0.2)0.0131}{e^{(0.12+0.15)}} = 0.0332$$

**Solution 64****B** Chapter 24, Black-Derman-Toy Model 

The ratio of the short rate to the rate below it is constant within a column, so:

$$\frac{r_{uu}}{r_{ud}} = \frac{r_{ud}}{r_{dd}} \quad \Rightarrow \quad \frac{0.45}{r_{ud}} = \frac{r_{ud}}{0.20} \quad \Rightarrow \quad r_{ud} = 0.30$$

The value of a 2-year bond is:

$$\begin{aligned} P(0,2) &= 0.5 \left[ \left( \frac{1}{1+r_0} \right) \left( \frac{1}{1+r_u} \right) + \left( \frac{1}{1+r_0} \right) \left( \frac{1}{1+r_d} \right) \right] \\ &= 0.5 \frac{1}{1+r_0} \left[ \frac{1}{1.40} + \frac{1}{1.25} \right] = \frac{0.75714}{1+r_0} \end{aligned}$$

The value of a 3-year bond is:


$$\begin{aligned} P(0,3) &= 0.25 \left[ \left( \frac{1}{1+r_0} \right) \left( \frac{1}{1+r_u} \right) \left( \frac{1}{1+r_{uu}} \right) + \left( \frac{1}{1+r_0} \right) \left( \frac{1}{1+r_u} \right) \left( \frac{1}{1+r_{ud}} \right) \right. \\ &\quad \left. + \left( \frac{1}{1+r_0} \right) \left( \frac{1}{1+r_d} \right) \left( \frac{1}{1+r_{ud}} \right) + \left( \frac{1}{1+r_0} \right) \left( \frac{1}{1+r_d} \right) \left( \frac{1}{1+r_{dd}} \right) \right] \\ &= 0.25 \frac{1}{1+r_0} \left[ \left( \frac{1}{1.40} \right) \left( \frac{1}{1.45} \right) + \left( \frac{1}{1.40} \right) \left( \frac{1}{1.30} \right) + \left( \frac{1}{1.25} \right) \left( \frac{1}{1.30} \right) + \left( \frac{1}{1.25} \right) \left( \frac{1}{1.20} \right) \right] \\ &= \frac{0.58103}{1+r_0} \end{aligned}$$

The forward price is:

$$F_{0,2}[P(2,3)] = P_0(2,3) = \frac{P(0,3)}{P(0,2)} = \frac{\frac{0.58103}{1+r_0}}{\frac{0.75714}{1+r_0}} = 0.76740$$

Multiplying by 1,000, we have:

$$1,000 \times F_{0,2}[P(2,3)] = 1,000 \times 0.76740 = 767.40$$

**Solution 65****C** Chapter 24, Risk-Neutral Version of the Vasicek Model 

The process for the short rate follows the Vasicek Model. The true process can be rewritten in the familiar form:

$$dr = 0.08(0.10 - r)dt + 0.04dZ \quad \Rightarrow \quad a = 0.08, b = 0.10, \sigma = 0.04$$

The risk-neutral process can be written as:

$$\begin{aligned} dr &= [\mathbf{a}(r) + \sigma(r)\phi(r,t)]dt + \sigma(r)d\tilde{Z} = [\mathbf{a}(b-r) + \sigma\phi]dt + \sigma d\tilde{Z} \\ &= [0.08(0.10-r) + 0.04\phi]dt + 0.04d\tilde{Z} \end{aligned}$$

We can use set the bracketed expression above equal to the bracketed expression in (ii):

$$\begin{aligned} 0.08(0.10-r) + 0.04\phi &= 0.0128 - 0.08r \\ 0.008 + 0.04\phi - 0.08r &= 0.0128 - 0.08r \\ \phi &= 0.12 \end{aligned}$$

The parameter  $B(2,6)$  is:

$$\begin{aligned} B(t,T) &= \frac{1 - e^{-a(T-t)}}{a} \\ B(2,6) &= \frac{1 - e^{-0.08(6-2)}}{0.08} = 3.4231 \end{aligned}$$

We can use the Sharpe ratio to determine  $\alpha(0.04,2,6)$ :

$$\begin{aligned} \phi(r,t) &= \frac{\alpha(r,t,T) - r}{q(r,t,T)} \\ \phi &= \frac{\alpha(r,t,T) - r}{B(t,T)\sigma} \\ \alpha(r,t,T) &= r + B(t,T)\sigma\phi \\ \alpha(0.04,2,6) &= 0.04 + 3.4231 \times 0.04 \times 0.12 = 0.05643 \end{aligned}$$

### Solution 66

**B** Chapter 24, Risk-Neutral Version of the Vasicek Model



From (i), we observe that:

$$\begin{aligned} \mathbf{a}(r) &= x - y \times r(t) \\ \sigma(r) &= \sigma = 0.12 \end{aligned}$$

The risk-neutral version of the Vasicek model is:

$$dr = [\mathbf{a}(r) + \sigma(r)\phi(r,t)]dt + \sigma(r)d\tilde{Z} = [x - y \times r(t) + 0.04\phi]dt + 0.04d\tilde{Z}$$

Setting the above expression in brackets equal to the expression in brackets from (ii), we have:

$$\begin{aligned} x - y \times r(t) + 0.04\phi &= x + 0.0048 - y \times r(t) \\ x + 0.04\phi &= x + 0.0048 \\ 0.04\phi &= 0.0048 \\ \phi &= 0.12 \end{aligned}$$

The parameter  $B(3,7)$  is:

$$B(t,T) = \frac{1 - e^{-a(T-t)}}{a}$$

$$B(3,7) = \frac{1 - e^{-a(7-3)}}{a} = \frac{1 - e^{-4y}}{y} = \bar{a}_{4|y} = 3.42314$$

We can use the Sharpe ratio to determine  $\alpha(0.06,3,7)$ :

$$\phi(r,t) = \frac{\alpha(r,t,T) - r}{q(r,t,T)}$$

$$\phi = \frac{\alpha(r,t,T) - r}{B(t,T)\sigma}$$

$$\alpha(r,t,T) = r + B(t,T)\sigma\phi$$

$$\alpha(0.06,3,7) = 0.06 + 3.42314 \times 0.04 \times 0.12 = 0.07643$$

### Solution 67

C Chapter 12, Risk-Neutral Version of the Vasicek Model



The process for the short rate follows the Vasicek Model. The true process can be rewritten in the familiar form:

$$dr = 0.08(0.10 - r)dt + 0.04dZ \quad \Rightarrow \quad a = 0.08, b = 0.10, \sigma = 0.04$$

We can use (iii) to obtain the Sharpe ratio:

$$\tilde{Z}(t) = Z(t) - \phi t$$

$$\tilde{Z}(4) = Z(4) - 4\phi$$

$$0.02 = 0.22 - 4\phi$$

$$\phi = 0.05$$

The parameter  $B(4,8)$  is:

$$B(t,T) = \frac{1 - e^{-a(T-t)}}{a}$$

$$B(4,8) = \frac{1 - e^{-a(8-4)}}{a} = \frac{1 - e^{-0.08 \times 4}}{0.08} = 3.42314$$

We can use the Sharpe ratio to determine  $\alpha(0.07,4,8)$ :

$$\phi(r,t) = \frac{\alpha(r,t,T) - r}{q(r,t,T)}$$

$$\phi = \frac{\alpha(r,t,T) - r}{B(t,T)\sigma}$$

$$\alpha(r,t,T) = r + B(t,T)\sigma\phi$$

$$\alpha(0.07,4,8) = 0.07 + 3.42314 \times 0.04 \times 0.05 = 0.0768$$

**Solution 68****E** Chapter 24, Cox-Ingersoll-Ross Model

The CIR model is:

$$dr = a(b - r)dt + \sigma\sqrt{r}dZ$$

From the partial differential equation provided in the question, we can obtain the following parameters for the CIR model:

$$a = 0.14 \qquad b = \frac{0.014}{0.14} = 0.10 \qquad \sigma = 0.12$$

We can define  $c$  in terms of  $\bar{\phi}$ :

$$\phi(r, t) = \bar{\phi} \frac{\sqrt{r}}{\sigma} \quad \& \quad \phi(r, t) = c\sqrt{r} \quad \Rightarrow \quad c = \frac{\bar{\phi}}{\sigma}$$

In the CIR model, as the maturity of a zero-coupon bond approaches infinity, its yield approaches:

$$\bar{r} = \frac{2ab}{(a - \bar{\phi} + \gamma)} = \lim_{T \rightarrow \infty} \left[ \frac{-\ln[P(r, t, T)]}{T - t} \right]$$

From the information provided in part (ii) of the question, we know that the rightmost portion is equal to 0.14. Therefore:

$$\begin{aligned} \frac{2ab}{(a - \bar{\phi} + \gamma)} &= 0.14 \\ \frac{2 \times 0.14 \times 0.10}{(0.14 - \bar{\phi} + \gamma)} &= 0.14 \\ 0.028 &= 0.14(0.14 - \bar{\phi} + \gamma) \\ 0.2 &= 0.14 - \bar{\phi} + \gamma \\ \gamma &= \bar{\phi} + 0.06 \end{aligned}$$


We can substitute this value into the formula for  $\gamma$ :

$$\begin{aligned} \gamma &= \sqrt{(a - \bar{\phi})^2 + 2\sigma^2} \\ \bar{\phi} + 0.06 &= \sqrt{(0.14 - \bar{\phi})^2 + 2(0.12)^2} \\ \bar{\phi}^2 + 0.12\bar{\phi} + 0.0036 &= (0.14 - \bar{\phi})^2 + 2(0.12)^2 \\ \bar{\phi}^2 + 0.12\bar{\phi} + 0.0036 &= (0.0196 - 0.28\bar{\phi} + \bar{\phi}^2) + 0.0288 \\ 0.4\bar{\phi} &= 0.0448 \\ \bar{\phi} &= 0.112 \end{aligned}$$

The value of  $c$  is:

$$c = \frac{\bar{\phi}}{\sigma} = \frac{0.112}{0.12} = 0.9333$$

**Solution 69**

**C** Chapter 24, Vasicek Model 

*The Vasicek model allows negative yields.*

The Vasicek model is:

$$dr = a(b - r)dt + \sigma dZ$$

From the partial differential equation provided in the question, we can obtain the following parameters for the CIR model:

$$a = 0.20 \quad b = \frac{0.01}{0.2} = 0.05 \quad \sigma = 0.14$$


In the Vasicek model, as the maturity of a zero-coupon bond approaches infinity, its yield approaches:

$$\bar{r} = b + \phi \frac{\sigma}{a} - 0.5 \frac{\sigma^2}{a^2} = \lim_{T \rightarrow \infty} \left[ \frac{-\ln[P(r, t, T)]}{T - t} \right]$$

From the information provided the question, we know that  $\bar{r}$  is equal to  $-0.083$ . Therefore:

$$\begin{aligned} b + \phi \frac{\sigma}{a} - 0.5 \frac{\sigma^2}{a^2} &= -0.083 \\ 0.05 + \phi \frac{0.14}{0.20} - 0.5 \frac{0.14^2}{0.20^2} &= -0.083 \\ \phi &= 0.16 \end{aligned}$$

**Solution 70**

**A** Chapter 24, Rendleman-Bartter Model 

In the Rendleman-Bartter Model, the short-term rate follows geometric Brownian motion. From the Chapter 18 Review Note, the payoff of a call option can be found as:

$$E[\text{Call Payoff}] = S_t e^{(\alpha - \delta)(T-t)} N(\hat{d}_1) - K \times N(\hat{d}_2)$$

We can consider the derivative to be 1,000 call options with the short rate replacing the stock price and 0.10 replacing the strike price. Although we have  $-0.25$  for the volatility parameter in the geometric Brownian motion, we use the absolute value of  $-0.25$  in the formulas for  $\hat{d}_1$  and  $\hat{d}_2$ , because the  $\sigma$  in the formulas is the standard deviation of the stock's return, and it must therefore be positive:

$$\hat{d}_1 = \frac{\ln\left(\frac{r(t)}{K}\right) + (\alpha - \delta + 0.5\sigma^2)(T-t)}{\sigma\sqrt{T-t}} = \frac{\ln\left(\frac{0.09}{0.10}\right) + (0.10 + 0.5 \times 0.25^2) \times 0.25}{0.25\sqrt{0.25}} = -0.58038$$

$$\hat{d}_2 = \frac{\ln\left(\frac{r(t)}{K}\right) + (\alpha - \delta - 0.5\sigma^2)(T-t)}{\sigma\sqrt{T-t}} = \frac{\ln\left(\frac{0.09}{0.10}\right) + (0.10 - 0.5 \times 0.25^2) \times 0.25}{0.25\sqrt{0.25}} = -0.70538$$

The values of  $N(\hat{d}_1)$  and  $N(\hat{d}_2)$  are:

$$N(\hat{d}_1) = N(-0.58038) = 0.28083$$

$$N(\hat{d}_2) = N(-0.70538) = 0.24029$$

The expected value of the payoff is:

$$E[\text{Payoff}] = 1,000 \times \left[ S_t e^{(\alpha - \delta)(T-t)} N(\hat{d}_1) - K \times N(\hat{d}_2) \right]$$

$$= 1,000 \times \left[ 0.09 e^{0.10(0.25)} \times 0.28083 - 0.10 \times 0.24029 \right] = 1.8855$$

### Solution 71

A Chapter 24, Black Model



The bond forward price is:

$$F = P_0(T, T+s) = \frac{P(0, T+s)}{P(0, T)} = \frac{P(0, 2)}{P(0, 1)} = \frac{0.8417}{0.9259} = 0.90906$$

The values of  $d_1$  and  $d_2$  are:

$$d_1 = \frac{\ln\left(\frac{F}{K}\right) + 0.5\sigma^2 T}{\sigma\sqrt{T}} = \frac{\ln\left(\frac{0.90906}{0.90009}\right) + 0.5(0.10)^2(1)}{0.10\sqrt{1}} = 0.14918$$

$$d_2 = d_1 - \sigma\sqrt{T} = 0.14918 - 0.10\sqrt{1} = 0.04918$$

We have:

$$N(-d_1) = N(-0.14918) = 0.44071$$

$$N(-d_2) = N(-0.04918) = 0.48039$$

The Black formula for the put price is:

$$\begin{aligned} P &= P(0, T) [K \times N(-d_2) - F \times N(-d_1)] \\ &= 0.9259 [0.90009 \times 0.48039 - 0.90906 \times 0.44071] \\ &= 0.029408 \end{aligned}$$

### Solution 72

**E** Chapter 24, Black Model



The price of the caplet is:

$$\text{Caplet price} = (1.111) \times (\text{Put Option Price})$$

The put option:

- has a zero-coupon bond that expires at time 2 as its underlying asset
- expires at time 1
- has a strike price of:

$$K = \frac{1}{1 + K_R} = \frac{1}{1.111} = 0.90009$$

The bond forward price is:

$$F = P_0(T, T + s) = \frac{P(0, T + s)}{P(0, T)} = \frac{P(0, 2)}{P(0, 1)} = \frac{0.8417}{0.9259} = 0.90906$$

The volatility of the bond forward price is:

$$\begin{aligned} \sigma &= \sqrt{\frac{\text{Var}\{\ln[P_t(T, T + s)]\}}{t}} = \sqrt{\frac{\text{Var}\{\ln[P_t(1, 2)]\}}{t}} = \sqrt{\frac{\text{Var}\left\{\ln\left[\frac{P(t, 2)}{P(t, 1)}\right]\right\}}{t}} \\ &= \sqrt{\frac{0.01t}{t}} = \sqrt{0.01} = 0.1 \end{aligned}$$

The values of  $d_1$  and  $d_2$  are:

$$\begin{aligned} d_1 &= \frac{\ln\left(\frac{F}{K}\right) + 0.5\sigma^2 T}{\sigma\sqrt{T}} = \frac{\ln\left(\frac{0.90906}{0.90009}\right) + 0.5(0.10)^2(1)}{0.10\sqrt{1}} = 0.14918 \\ d_2 &= d_1 - \sigma\sqrt{T} = 0.14918 - 0.10\sqrt{1} = 0.04918 \end{aligned}$$

We have:

$$N(-d_1) = N(-0.14918) = 0.44071$$

$$N(-d_2) = N(-0.04918) = 0.48039$$

The Black formula for the put price is:

$$\begin{aligned} P &= P(0, T) [K \times N(-d_2) - F \times N(-d_1)] \\ &= 0.9259 [0.90009 \times 0.48039 - 0.90906 \times 0.44071] \\ &= 0.029408 \end{aligned}$$

The price of the caplet is:

$$\text{Caplet price} = (1.111) \times (\text{Put Option Price}) = 1.111 \times 0.029408 = 0.03267$$

### Solution 73

**D** Chapter 24, Cox-Ingersoll-Ross Model



In the CIR Model, we have:

$$P(r, t, T) = A(t, T)e^{-B(t, T)r}$$

In the CIR model, we have:

$$A(t, T) = A(0, T - t) \text{ and } B(t, T) = B(0, T - t).$$

This implies that we can define  $A$  and  $B$  as shown below:

$$\begin{aligned} A &= A(0, 8) = A(1, 9) = A(3, 11) \\ B &= B(0, 8) = B(1, 9) = B(3, 11) \end{aligned}$$

We can obtain the ratio of the second price to the first price:

$$\left. \begin{aligned} P(r, 0, 8) &= 0.4005 = Ae^{-Br} \\ P(1.25r, 1, 9) &= 0.3581 = Ae^{-1.25Br} \end{aligned} \right\} \Rightarrow e^{-0.25Br} = \frac{0.3581}{0.4005}$$

The ratio obtained above can be used to find the bond price needed to answer the question:

$$P(1.5r, 3, 11) = Ae^{-1.5Br} = Ae^{-1.25r} \times e^{-0.25r} = 0.3581 \times \frac{0.3581}{0.4005} = 0.3202$$