Chapter 11 – Solutions

Solution 1

C Chapter 11, State Prices

The question tells us that the stock price might decrease to $*X*, which implies that:

$$X < 100$$

The stock price tree and the call option tree are:

Stock		Call	
	150.0000		50.0000
100.0000		23.7200	
	X		0.0000

We can use the price of the call option to obtain the state price associated with an upward movement:

$$23.72 = 50Q_u$$
$$Q_u = 0.4744$$

The risk-free rate of return can now be used to obtain the state price associated with a downward movement:

$$e^{-rh} = Q_u + Q_d$$
$$e^{-0.10\times 1} = 0.4744 + Q_d$$
$$Q_d = 0.4304$$

We can use the stock's price and the state prices to obtain the value of X:

$$100 = 150Q_u + XQ_d$$
$$100 = 150 \times 0.4744 + 0.4304 \times X$$
$$X = 67.0016$$

The strike price of the put option is:

$$K = 1.6X = 1.6 \times 67.0016 = 107.2026$$

The put option pays off only if the stock's price falls to 67.0016, so the current value of the put option is:

$$0Q_u + (107.2026 - 67.0016)Q_d = 0 + 40.2010 \times 0.4304 = 17.3040$$

<u>Alternate Solution</u>

We don't have to use state prices to answer this question.

Since the call option has a payoff of 50 if the stock price increases and 0 if the stock price falls, we can use the call price to obtain the risk-neutral probability of an upward movement:

$$23.72 = e^{-0.10}\left[50p^* + 0(1-p^*)\right]$$
$$p^* = 0.5243$$

We can now use the current stock price to solve for X:

$$100 = e^{-0.10}[150p^* + X(1-p^*)]$$
$$100 = e^{-0.10}[150\times 0.5243 + X(1-0.5243)]$$
$$X = 67.0016$$

The strike price of the put option is:

$$K = 1.6X = 1.6\times 67.0016 = 107.2026$$

The put option pays off only if the stock price falls to 67.0016, so the current value of the put option is:

$$e^{-0.10}\left[0\times 0.5243 + (107.2026 - 67.0016)(1-0.5243)\right] = 17.3040$$

Solution 2

D Chapter 11, Early Exercise of Perpetual Options

The option should be exercised early if and only if:

$$\delta S > rK$$

This is equivalent to:

$$S > \frac{rK}{\delta}$$

This condition is satisfied when:

$$S > \frac{(0.14)(30)}{0.11}$$
$$S > 38.1818$$

Therefore, the lowest price (rounded up to the nearest penny) for which early exercise is optimal is $38.19.

Solution 3

C Chapter 11, Early Exercise of Perpetual Options

The option should be exercised early if and only if:

$$S > \frac{rK}{\delta}$$

This condition is satisfied when:

$$S > \frac{(0.07)(68)}{0.05}$$
$$S > 95.20$$

The option is called when the stock price reaches $95.20.

The volatility of the stock is 0%, so the return on the stock is the risk-free rate of return. The stock pays dividends at a rate of 5%, so for it to earn the risk-free rate of 7% it must increase in price at a rate of 2%:

$$r - \delta = 0.07 - 0.05 = 0.02$$

Another way to see this is to consider that in the binomial model, u and d are equal:

$$u = e^{(r-\delta)h+\sigma\sqrt{h}} = e^{(0.07-0.05)h+0.00\sqrt{h}} = e^{0.02h}$$
$$d = e^{(r-\delta)h-\sigma\sqrt{h}} = e^{(0.07-0.05)h-0.00\sqrt{h}} = e^{0.02h}$$

Regardless of the size of h, the stock price increases at a continuously compounded rate of 2% per year.

The time until the stock price reaches $95.20 can be found by solving the following equation for T:

$$70e^{0.02T} = 95.20$$
$$e^{0.02T} = \frac{95.20}{70}$$
$$0.02T = \ln\left(\frac{95.20}{70}\right)$$
$$T = \ln\left(\frac{95.20}{70}\right) \times \frac{1}{0.02}$$
$$T = 15.37$$

Solution 4

B Chapter 11, Early Exercise of Perpetual Options

The option should be exercised early if and only if:

$$S > \frac{rK}{\delta}$$

This condition is satisfied when:

$$S > \frac{(0.09)(48)}{0.06}$$
$$S > 72.00$$

The option is called when the stock price reaches $72.00.

The volatility of the stock is 0%, so the return on the stock is the risk-free rate of return. The stock pays dividends at a rate of 6%, so for it to earn the risk-free rate of 9% it must increase in price at a rate of 3%:

$$r - \delta = 0.09 - 0.06 = 0.03$$

The time until the stock price reaches $72.00 can be found by solving the following equation for T:

$$50e^{0.03T} = 72$$

$$0.03T = \ln\left(\frac{72}{50}\right)$$

$$T = \ln\left(\frac{72}{50}\right) \times \frac{1}{0.03}$$

$$T = 12.1548$$

The option is exercised in 12.1548 years when the stock price is $72.00. At that time, the exercise value is:

$$72 - 48 = 24$$

Since the volatility is zero, the American call option is certain to pay out $24 in 12.1548 years. Since the payoff is certain, its present value is obtained by discounting at the risk-free rate of return:

$$24e^{-12.1548r} = 24e^{-12.1548(0.09)} = 8.0376$$

Solution 5

C Chapter 11, Exercise Boundaries

Answer A is not correct because an increase in the volatility of the stock increases the value of the implicit insurance, thereby making early exercise less attractive.

Answer B is not correct because an increase in the volatility of the stock increases the value of the implicit insurance, thereby making early exercise less attractive.

Answer C is correct because an increase in the volatility of the stock increases the value of the implicit insurance, making early exercise less attractive and increasing the exercise boundary that must be reached in order for early exercise to be optimal.

Answer D is not correct because as an American call option ages, the early-exercise criteria become less stringent, which means that the exercise boundary decreases.

Answer E is not correct because as an American put option ages, the early-exercise criteria become less stringent, which means that the exercise boundary increases.

Solution 6

E Chapter 11, Exercise Boundaries

For each option, there are 3 effects to consider: the change in volatility, the new stock price, and the passage of time.

A call option is exercised only if the stock's price is greater than the exercise boundary, and a put option is exercised only if the stock's price is less than the exercise boundary.

Let's consider the options in order.

American call option on Stock X

Higher volatility increases the exercise boundary:	Less likely exercise
Unchanged price does not affect likelihood of exercise:	No effect
Passage of time decreases the exercise boundary:	More likely exercise
Net effect:	Indeterminate

American call option on Stock Y

Lower volatility decreases the exercise boundary:	More likely exercise
Lower price decreases the likelihood of exercise:	Less likely exercise
Passage of time decreases the exercise boundary:	More likely exercise
Net effect:	Indeterminate

American call option on Stock Z

Higher volatility increases the exercise boundary:	Less likely exercise
Lower price decreases the likelihood of exercise:	Less likely exercise
Passage of time decreases the exercise boundary:	More likely exercise
Net effect:	Indeterminate

American put option on Stock X

Higher volatility decreases the exercise boundary:	Less likely exercise
Unchanged price does not affect likelihood of exercise:	No effect
Passage of time increases the exercise boundary:	More likely exercise
Net effect:	Indeterminate

American put option on Stock Y

Lower volatility increases the exercise boundary:	More likely exercise
Lower price increases the likelihood of exercise:	More likely exercise
Passage of time increases the exercise boundary:	More likely exercise
Net effect:	More likely exercise

The only option which we can say for sure is more likely to be exercised now is the American put option on Stock Y. Since we are told that only one of the options is optimal to exercise now, that option must be the put option on Stock Y.

Solution 7

D Chapter 11, Risk-Neutral Pricing

The values of u and d are:

$$u = e^{(r-\delta)h+\sigma\sqrt{h}} = e^{(0.08-0.00)(1)+0.30\sqrt{1}} = 1.46228$$
$$d = e^{(r-\delta)h-\sigma\sqrt{h}} = e^{(0.08-0.00)(1)-0.30\sqrt{1}} = 0.80252$$

The possible stock prices at the end of one year are:

$$50u = 50 \times 1.46228 = 73.1142$$
$$50d = 50 \times 0.80252 = 40.1259$$

The payoffs for the call option in 1 year are:

Up state: $Max[0,\ 73.1142 - 48] = 25.1142$
Down state: $Max[0,\ 40.1259 - 48] = 0.0000$

The stock price tree and the call price tree are:

Stock		Call	
	73.1142		25.1142
50.0000		V	
	40.1259		0.0000

The risk-neutral probability of an upward movement is:

$$p^* = \frac{e^{(r-\delta)h} - d}{u-d} = \frac{e^{(0.08-0.00)(1)} - 0.80252}{1.46228 - 0.80252} = 0.42556$$

The value of the call option is:

$$V = e^{-rh}\left[(p^*)V_u + (1-p^*)V_d\right] = e^{-0.08(1)}\left[0.42556(25.1142) + (1-0.42556)(0.0000)\right]$$
$$= 9.8659$$

Solution 8

D Chapter 11, Expected Return

The values of u and d are:

$$u = e^{(r-\delta)h+\sigma\sqrt{h}} = e^{(0.08-0.00)(1)+0.30\sqrt{1}} = 1.46228$$
$$d = e^{(r-\delta)h-\sigma\sqrt{h}} = e^{(0.08-0.00)(1)-0.30\sqrt{1}} = 0.80252$$

The stock price tree and the call price tree are:

Stock		Call	
	73.1142		25.1142
50.0000		V	
	40.1259		0.0000

The risk-neutral probability of an upward movement is:

$$p^* = \frac{e^{(r-\delta)h} - d}{u-d} = \frac{e^{(0.08-0.00)(1)} - 0.80252}{1.46228 - 0.80252} = 0.42556$$

The value of the call option is:

$$V = e^{-rh}\left[(p^*)V_u + (1-p^*)V_d\right] = e^{-0.08(1)}\left[0.42556(25.1142) + (1-0.42556)(0.0000)\right]$$
$$= 9.8659$$

Since we know the price of the option and the distribution of its payoffs, we can now found the expected return of the option:

$$V = e^{-\gamma h}\left[(p)V_u + (1-p)V_d\right]$$

$$9.8659 = e^{-\gamma}\left[0.46 \times 25.1142 + (1-0.46) \times 0.0000\right]$$

$$\gamma = -\ln\left(\frac{9.8659}{0.46 \times 25.1142}\right) = 0.15783$$

Solution 9

C Chapter 11, Expected Return

The return on a put option is generally less than the risk-free rate of return. For a put option, Δ is negative and B is positive. This means that buying a put option is equivalent to selling stock and lending at the risk-free rate. The expected return is less than the risk-free rate since the stock's expected return is greater than the risk-free rate.

The values of u and d are:

$$u = e^{(r-\delta)h + \sigma\sqrt{h}} = e^{(0.08-0.00)(1)+0.30\sqrt{1}} = 1.46228$$

$$d = e^{(r-\delta)h - \sigma\sqrt{h}} = e^{(0.08-0.00)(1)-0.30\sqrt{1}} = 0.80252$$

The stock price tree and the put price tree are:

Stock		Put	
	73.1142		0.0000
50.0000		V	
	40.1259		7.8741

The risk-neutral probability of an upward movement is:

$$p^* = \frac{e^{(r-\delta)h} - d}{u-d} = \frac{e^{(0.08-0.00)(1)} - 0.80252}{1.46228 - 0.80252} = 0.42556$$

The value of the put option is:

$$V = e^{-rh}\left[(p^*)V_u + (1-p^*)V_d\right] = e^{-0.08(1)}\left[0.42556(0.0000) + (1-0.42556)(7.8741)\right]$$
$$= 4.1754$$

Since we know the price of the option and the distribution of its payoffs, we can now found the expected return of the option:

$$V = e^{-\gamma h}\left[(p)V_u + (1-p)V_d\right]$$

$$4.1754 = e^{-\gamma}\left[0.46\times 0.0000 + (1-0.46)\times 7.8741\right]$$

$$\gamma = -\ln\left(\frac{4.1754}{(1-0.46)\times 7.8741}\right) = 0.01817$$

Solution 10

A Chapter 11, Expected Return

The values of u and d are:

$$u = e^{(r-\delta)h+\sigma\sqrt{h}} = e^{(0.08-0.04)(1)+0.24\sqrt{1}} = 1.32313$$

$$d = e^{(r-\delta)h-\sigma\sqrt{h}} = e^{(0.08-0.04)(1)-0.24\sqrt{1}} = 0.81873$$

The stock price tree and the put price tree are below:

Stock		Put	
	82.0340		0.0000
62.0000		V	
	50.7613		13.2387

The realistic probability of an upward movement is:

$$p = \frac{e^{(\alpha-\delta)h} - d}{u-d} = \frac{e^{(0.12-0.04)(1)} - 0.81873}{1.32313 - 0.81873} = 0.52450$$

The risk-neutral probability of an upward movement is:

$$p^* = \frac{e^{(r-\delta)h} - d}{u-d} = \frac{e^{(0.08-0.04)(1)} - 0.81873}{1.32313 - 0.81873} = 0.44029$$

The value of the put option can be calculated using the risk-neutral probability:

$$V = e^{-rh}\left[(p^*)V_u + (1-p^*)V_d\right] = e^{-0.08(1)}\left[0.44029(0.0000) + (1-0.44029)(13.2387)\right]$$
$$= 6.8402$$

Since we know the price of the option and the distribution of its payoffs, we can now found the expected return of the option:

$$V = e^{-\gamma h}\left[(p)V_u + (1-p)V_d\right]$$

$$6.8402 = e^{-\gamma}\left[0.52450\times 0.0000 + (1-0.52450)\times 13.2387\right]$$

$$\gamma = -\ln\left(\frac{6.8402}{(1-0.52450)\times 13.2387}\right) = -0.08305$$

Solution 11

B Chapter 11, Realistic Probability

The values of u and d are:

$$u = e^{(r-\delta)h+\sigma\sqrt{h}} = e^{(0.07-0.02)(1)+0.27\sqrt{1}} = 1.37713$$
$$d = e^{(r-\delta)h-\sigma\sqrt{h}} = e^{(0.07-0.02)(1)-0.27\sqrt{1}} = 0.80252$$

The stock price tree and the call price tree are:

Stock		Call	
	52.3309		12.3309
38.0000		V	
	30.4957		0.0000

The risk-neutral probability of an upward movement is:

$$p^* = \frac{e^{(r-\delta)h} - d}{u-d} = \frac{e^{(0.07-0.02)(1)} - 0.80252}{1.37713 - 0.80252} = 0.43291$$

The value of the call option is:

$$V = e^{-rh}\left[(p^*)V_u + (1-p^*)V_d\right] = e^{-0.07(1)}\left[0.43291(12.3309) + (1-0.43291)(0.0000)\right]$$
$$= 4.9772$$

Since we know the price of the option, its possible payoffs, and its expected rate of return, we can find the realistic probability of an upward movement:

$$V = e^{-\gamma h}\left[(p)V_u + (1-p)V_d\right]$$
$$4.9772 = e^{-0.34836(1)}\left[p \times 12.3309 + (1-p) \times 0.0000\right]$$
$$p = \frac{4.9772 \times e^{0.34836}}{12.3309} = 0.57185$$

The realistic probability that the stock price moves down is:

$$1 - 0.57185 = 0.42815$$

Solution 12

C Chapter 11, Expected Return

The values of u and d are:

$$u = e^{(r-\delta)h+\sigma\sqrt{h}} = e^{(0.07-0.02)(1)+0.27\sqrt{1}} = 1.37713$$
$$d = e^{(r-\delta)h-\sigma\sqrt{h}} = e^{(0.07-0.02)(1)-0.27\sqrt{1}} = 0.80252$$

The stock price tree and the call price tree are:

Stock		Call	
	52.3309		12.3309
38.0000		V	
	30.4957		0.0000

The risk-neutral probability of an upward movement is:

$$p^* = \frac{e^{(r-\delta)h} - d}{u-d} = \frac{e^{(0.07-0.02)(1)} - 0.80252}{1.37713 - 0.80252} = 0.43291$$

The value of the call option is:

$$V = e^{-rh}\left[(p^*)V_u + (1-p^*)V_d\right] = e^{-0.07(1)}\left[0.43291(12.3309) + (1-0.43291)(0.0000)\right]$$
$$= 4.9772$$

Since we know the price of the option, its possible payoffs, and its expected rate of return, we can find the realistic probability of an upward movement:

$$V = e^{-\gamma h}\left[(p)V_u + (1-p)V_d\right]$$
$$4.9772 = e^{-0.34836(1)}\left[p \times 12.3309 + (1-p) \times 0.0000\right]$$
$$p = \frac{4.9772 \times e^{0.34836}}{12.3309} = 0.57185$$

We can now use the realistic probability to find the expected return on the stock:

$$p = \frac{e^{(\alpha-\delta)h} - d}{u-d}$$
$$0.57185 = \frac{e^{(\alpha-0.02)\times 1} - 0.80252}{1.37713 - 0.80252}$$
$$\alpha = 0.1432$$

Solution 13

B Chapter 11, Expected Return

The realistic probability of an up jump is constant throughout the tree. The expected return on the stock is also constant throughout the tree. The return on the option, however, can change from one node to the next.

The values of u and d are:

$$u = e^{(r-\delta)h + \sigma\sqrt{h}} = e^{(0.06-0.05)(0.5)+0.30\sqrt{0.5}} = 1.24251$$
$$d = e^{(r-\delta)h - \sigma\sqrt{h}} = e^{(0.06-0.05)(0.5)-0.30\sqrt{0.5}} = 0.81291$$

The risk-neutral probability of an upward movement is:

$$p^* = \frac{e^{(r-\delta)h} - d}{u-d} = \frac{e^{(0.06-0.05)(0.5)} - 0.81291}{1.24251 - 0.81291} = 0.44716$$

The stock price tree and the call price tree are shown below:

Stock			European Call		
		77.1913			30.1913
	62.1254			14.9806	
50.0000		50.5025	7.3162		3.5025
	40.6456			1.5199	
		33.0413			0.0000

Although the entire tree is shown above, it is not necessary to calculate the values along the lowest path in order to answer this problem.

The realistic probability of an up jump is:

$$p = \frac{e^{(\alpha-\delta)h} - d}{u-d} = \frac{e^{(0.10-0.05)(0.5)} - 0.81291}{1.24251 - 0.81291} = 0.49442$$

If the stock price increases during the first 6 months, then the expected rate of return during the second 6 months must satisfy the following equation:

$$V = e^{-\gamma h}\left[(p)V_u + (1-p)V_d\right]$$

$$14.9806 = e^{-0.5\gamma}\left[0.49442 \times 30.1913 + (1-0.49442) \times 3.5025\right]$$

$$\gamma = \ln\left(\frac{0.49442 \times 30.1913 + (1-0.49442) \times 3.5025}{14.9806}\right) \times \frac{1}{0.5} = 0.21708$$

Solution 14

C Chapter 11, Expected Return

The values of u and d are:

$$u = e^{(r-\delta)h + \sigma\sqrt{h}} = e^{(0.06-0.05)(0.5) + 0.30\sqrt{0.5}} = 1.24251$$

$$d = e^{(r-\delta)h - \sigma\sqrt{h}} = e^{(0.06-0.05)(0.5) - 0.30\sqrt{0.5}} = 0.81291$$

The risk-neutral probability of an upward movement is:

$$p^* = \frac{e^{(r-\delta)h} - d}{u-d} = \frac{e^{(0.06-0.05)(0.5)} - 0.81291}{1.24251 - 0.81291} = 0.44716$$

The stock price tree and the call price tree are shown below:

Stock			American Put		
		77.1913			0.0000
	62.1254			0.0000	
50.0000		50.5025	4.0177		0.0000
	40.6456			7.4888	
		33.0413			13.9587

Although the entire tree is shown above, it is not necessary to calculate the values along the uppermost path in order to answer this problem.

Early exercise of the American put option is never optimal.

The realistic probability of an up-move is:

$$p = \frac{e^{(\alpha-\delta)h} - d}{u-d} = \frac{e^{(0.10-0.05)(0.5)} - 0.81291}{1.24251 - 0.81291} = 0.49442$$

If the stock price decreases during the first 6 months, then the expected rate of return during the second 6 months must satisfy the following equation:

$$V = e^{-\gamma h}\left[(p)V_u + (1-p)V_d\right]$$

$$7.4888 = e^{-0.5\gamma}\left[0.49442 \times 0.0000 + (1-0.49442) \times 13.9587\right]$$

$$\gamma = \ln\left(\frac{(1-0.49442)\times 13.9587}{7.4888}\right) \times \frac{1}{0.5} = -0.1187$$

Solution 15

E Chapter 11, Expected Return

The values of u and d are:

$$u = e^{(r-\delta)h + \sigma\sqrt{h}} = e^{(0.11-0.10)(1.0)+0.40\sqrt{1}} = 1.50682$$

$$d = e^{(r-\delta)h - \sigma\sqrt{h}} = e^{(0.11-0.10)(1.0)-0.40\sqrt{1}} = 0.67706$$

The risk-neutral probability of an upward movement is:

$$p^* = \frac{e^{(r-\delta)h} - d}{u-d} = \frac{e^{(0.11-0.10)(1.0)} - 0.67706}{1.50682 - 0.67706} = 0.40131$$

The stock price tree and the call price tree are shown below:

Stock			American Call		
		113.5250			73.5250
	75.3409			**35.3409**	
50.0000		51.0101	14.8283		11.0101
	33.8528			3.9582	
		22.9203			0.0000

If the stock moves up during the first year, then it is optimal to exercise the American option early. This is indicated above by the bold type for the option price of $35.3409.

The realistic probability of an up-move is:

$$p = \frac{e^{(\alpha-\delta)h} - d}{u-d} = \frac{e^{(0.24-0.10)(1.0)} - 0.67706}{1.50682 - 0.67706} = 0.57031$$

The expected return on the call option can be calculated using the realistic probability:

$$V = e^{-\gamma h}\left[(p)V_u + (1-p)V_d\right]$$

$$14.8283 = e^{-\gamma(1)}\left[0.57031 \times 35.3409 + (1-0.57031) \times 3.9582\right]$$

$$\gamma = \ln\left(\frac{0.57031 \times 35.3409 + (1-0.57031) \times 3.9582}{14.8283}\right) = 0.3879$$

Solution 16

D Chapter 11, Expected Return

The magnitude of γ decreases as the stock price increases, so we only need to check the top nodes to answer this question.

The risk-neutral probability of an upward movement is constant throughout the tree for the standard binomial model. Therefore we can calculate it using any node. Below we use the first node:

$$p^* = \frac{e^{(r-\delta)h} - d}{u-d} = \frac{e^{(0.08-0.00)(1/3)} - \dfrac{54.4126}{63.000}}{\dfrac{76.9385}{63.000} - \dfrac{54.4126}{63.000}} = 0.45681$$

Working from right to left, we create the tree of prices for the American call option:

Stock				American Call			
			114.7493				55.7493
		93.9609				36.5134	
	76.9385		81.1533		21.4521		22.1533
63.0000		66.4512		11.8596		9.8535	
	54.4126		57.3934		4.3827		0.0000
		46.9958				0.0000	
			40.5899				0.0000

Since the stock does not pay dividends, it is never optimal to exercise the call early.

The realistic probability is also constant throughout the tree, so as with the risk-neutral probability, we can use the values at the first node to calculate it:

$$p = \frac{e^{(\alpha-\delta)h} - d}{u-d} = \frac{e^{(0.15-0.00)(1/3)} - \frac{54.4126}{63.000}}{\frac{76.9385}{63.000} - \frac{54.4126}{63.000}} = 0.52462$$

Since γ decreases as the stock price increases, we need only check the uppermost branch of the tree of prices to find the lowest value of γ.

When the stock price is \$63.00, we have:

$$e^{\gamma h} = \frac{\left[(p)V_u + (1-p)V_d\right]}{V}$$

$$e^{\gamma(1/3)} = \frac{(0.52462)21.4521 + (1-0.52462)(4.3827)}{11.8596}$$

$$\gamma = \ln\left(\frac{(0.52462)21.4521 + (1-0.52462)(4.3827)}{11.8596}\right) \times 3 = 0.352$$

When the stock price is \$76.9385, we have:

$$e^{\gamma h} = \frac{\left[(p)V_u + (1-p)V_d\right]}{V}$$

$$e^{\gamma(1/3)} = \frac{(0.52462)36.5134 + (1-0.52462)(9.8535)}{21.4521}$$

$$\gamma = \ln\left(\frac{(0.52462)36.5134 + (1-0.52462)(9.8535)}{21.4521}\right) \times 3 = 0.317$$

When the stock price is \$93.9609, we have:

$$e^{\gamma h} = \frac{\left[(p)V_u + (1-p)V_d\right]}{V}$$

$$e^{\gamma(1/3)} = \frac{(0.52462)55.7493 + (1-0.52462)(22.1533)}{36.5134}$$

$$\gamma = \ln\left(\frac{(0.52462)55.7493 + (1-0.52462)(22.1533)}{36.5134}\right) \times 3 = 0.257$$

The lowest value of γ occurs when the stock price is \$93.9609.

Solution 17

D Chapter 11, Expected Return

The risk-neutral probability of an upward movement is constant throughout the tree for the standard binomial model. Therefore we can calculate it using any node. Below we use the first node:

$$p^* = \frac{e^{(r-\delta)h} - d}{u-d} = \frac{e^{(0.08-0.00)(1/3)} - \frac{54.4126}{63.000}}{\frac{76.9385}{63.000} - \frac{54.4126}{63.000}} = 0.45681$$

Working from right to left, we create the tree of prices for the European put option:

Stock				European Put			
			114.7493				0.0000
		93.9609				0.0000	
	76.9385		81.1533		0.4494		0.0000
63.0000		66.4512		3.3235		0.8497	
	54.4126		57.3934		5.9058		1.6066
		46.9958				10.4517	
			40.5899				18.4101

The realistic probability is also constant throughout the tree, so as with the risk-neutral probability, we can use the values at the first node to calculate it:

$$p = \frac{e^{(\alpha-\delta)h} - d}{u-d} = \frac{e^{(0.15-0.00)(1/3)} - \frac{54.4126}{63.000}}{\frac{76.9385}{63.000} - \frac{54.4126}{63.000}} = 0.52462$$

Since γ decreases as the stock price increases, we need only check the uppermost branch of the tree of prices to find the lowest value of γ.

When the stock price is \$63.00, we have:

$$e^{\gamma h} = \frac{\left[(p)V_u + (1-p)V_d\right]}{V}$$

$$e^{\gamma(1/3)} = \frac{(0.52462)0.4494 + (1-0.52462)(5.9058)}{3.3235}$$

$$\gamma = \ln\left(\frac{(0.52462)0.4494 + (1-0.52462)(5.9058)}{3.3235}\right) \times 3 = -0.264$$

When the stock price is \$76.9385, we have:

$$e^{\gamma h} = \frac{\left[(p)V_u + (1-p)V_d\right]}{V}$$

$$e^{\gamma(1/3)} = \frac{(0.52462)(0.0000) + (1-0.52462)(0.8497)}{0.4494}$$

$$\gamma = \ln\left(\frac{(0.52462)(0.0000) + (1-0.52462)(0.8497)}{0.4494}\right) \times 3 = -0.320$$

When the stock price is \$54.4126, we have:

$$e^{\gamma h} = \frac{\left[(p)V_u + (1-p)V_d\right]}{V}$$

$$e^{\gamma(1/3)} = \frac{(0.52462)(0.8497) + (1-0.52462)(10.4517)}{5.9058}$$

$$\gamma = \ln\left(\frac{(0.52462)(0.8497) + (1-0.52462)(10.4517)}{5.9058}\right) \times 3 = -0.261$$

When the stock price is \$66.4512 we have:

$$e^{\gamma h} = \frac{\left[(p)V_u + (1-p)V_d\right]}{V}$$

$$e^{\gamma(1/3)} = \frac{(0.52462)(0.0000) + (1-0.52462)(1.6066)}{0.8497}$$

$$\gamma = \ln\left(\frac{(0.52462)(0.0000) + (1-0.52462)(1.6066)}{0.8497}\right) \times 3 = -0.320$$

When the stock price is \$46.9958 we have:

$$e^{\gamma h} = \frac{\left[(p)V_u + (1-p)V_d\right]}{V}$$

$$e^{\gamma(1/3)} = \frac{(0.52462)(1.6066) + (1-0.52462)(18.4101)}{10.4517}$$

$$\gamma = \ln\left(\frac{(0.52462)(1.6066) + (1-0.52462)(18.4101)}{10.4517}\right) \times 3 = -0.257$$

The tree of values for γ is:

		N/A
	–0.320	
–0.264		–0.320
	–0.261	
		–0.257

The lowest value is –32.0%.

It might seem surprising that γ is lowest at the upper nodes. After all, the put option can be viewed as being more risky as it becomes more out-of-the money. So as the stock price increases, shouldn't the expected return increase as well? The answer is no, because the put option is a hedging instrument. Therefore, its "risk" is inversely related to the risk of the stock price declining. That's why put options earn less than the risk-free rate of return. Armed with this knowledge, we see that the lowest value of γ is sure to be in one of the upper nodes. Therefore, we could have solved this problem by calculating the value of γ at only the three uppermost nodes.

The general rule for both calls and puts is that the lowest value of γ for each column in a tree is at the highest node.

Solution 18

C Chapter 11, Cox-Ross-Rubinstein Binomial Tree

The values of u and d are:

$$u = e^{\sigma\sqrt{h}} = e^{0.30\sqrt{1}} = 1.34986$$
$$d = e^{-\sigma\sqrt{h}} = e^{-0.30\sqrt{1}} = 0.74082$$

The possible stock prices at the end of one year are:

$$38u = 38 \times 1.34986 = 51.2946$$
$$38d = 38 \times 0.74082 = 28.1511$$

The payoffs for the call option in 1 year are:

Up state: $Max[0,\ 51.2946 - 40] = 11.2946$

Down state: $Max[0,\ 28.1511 - 40] = 0.0000$

The stock price tree and the call price tree are below:

Stock		Call	
	51.2946		11.2946
38.0000		V	
	28.1511		0.0000

The risk-neutral probability of an upward movement is:

$$p^* = \frac{e^{(r-\delta)h} - d}{u-d} = \frac{e^{(0.07-0.00)(1)} - 0.74082}{1.34986 - 0.74082} = 0.54461$$

The value of the call option is:

$$V = e^{-rh}\left[(p^*)V_u + (1-p^*)V_d\right] = e^{-0.07(1)}\left[0.54461(11.2946) + (1-0.54461)(0.0000)\right]$$
$$= 5.735$$

Solution 19

A Chapter 11, Cox-Ross-Rubinstein Binomial Tree

The values of u and d are:

$$u = e^{\sigma\sqrt{h}} = e^{0.30\sqrt{1}} = 1.34986$$
$$d = e^{-\sigma\sqrt{h}} = e^{-0.30\sqrt{1}} = 0.74082$$

We can now solve for the expected return on the stock:

$$p = \frac{e^{(\alpha-\delta)h} - d}{u-d}$$
$$0.661 = \frac{e^{(\alpha-0)\times 1} - 0.74082}{1.34986 - 0.74082}$$
$$\alpha = 0.1340$$

Solution 20

B Chapter 11, Cox-Ross-Rubinstein Binomial Tree

The values of u and d are:

$$u = e^{\sigma\sqrt{h}} = e^{0.30\sqrt{1}} = 1.34986$$
$$d = e^{-\sigma\sqrt{h}} = e^{-0.30\sqrt{1}} = 0.74082$$

The possible stock prices at the end of one year are:

$$38u = 38 \times 1.34986 = 51.2946$$
$$38d = 38 \times 0.74082 = 28.1511$$

The payoffs for the put option in 1 year are:

Up state: $Max[0,\ 40 - 51.2946] = 0.0000$

Down state: $Max[0,\ 40 - 28.1511] = 11.8489$

The stock price tree and the put price tree are below:

Stock		Put	
	51.2946		0.0000
38.0000		V	
	28.1511		11.8489

The risk-neutral probability of an upward movement is:

$$p^* = \frac{e^{(r-\delta)h} - d}{u-d} = \frac{e^{(0.07-0.00)(1)} - 0.74082}{1.34986 - 0.74082} = 0.54461$$

The value of the put option is:

$$V = e^{-rh}\left[(p^*)V_u + (1-p^*)V_d\right] = e^{-0.07(1)}\left[0.54461(0.0000) + (1-0.54461)(11.8489)\right]$$
$$= 5.031$$

Solution 21

A Chapter 11, Cox-Ross-Rubinstein Binomial Tree

The values of u and d are:

$$u = e^{\sigma\sqrt{h}} = e^{0.30\sqrt{0.25}} = 1.16183$$
$$d = e^{-\sigma\sqrt{h}} = e^{-0.30\sqrt{0.25}} = 0.86071$$

The stock price tree and the call price tree are below:

Stock		Call	
	116.1834		21.1834
100.0000		10.1239	
	86.0708		0.0000

The risk-neutral probability of an upward movement is:

$$p^* = \frac{e^{(r-\delta)h} - d}{u-d} = \frac{e^{(0.08-0.05)(0.25)} - 0.86071}{1.16183 - 0.86071} = 0.48757$$

The value of the call option is:

$$V = e^{-rh}\left[(p^*)V_u + (1-p^*)V_d\right] = e^{-0.08(0.25)}\left[0.48757(21.1834) + (1-0.48757)(0.00)\right]$$
$$= 10.124$$

Solution 22

B Chapter 11, Cox-Ross-Rubinstein Binomial Tree

The values of u and d are:

$$u = e^{\sigma\sqrt{h}} = e^{0.30\sqrt{0.25}} = 1.16183$$
$$d = e^{-\sigma\sqrt{h}} = e^{-0.30\sqrt{0.25}} = 0.86071$$

The stock price tree and the call price tree are below:

Stock		Call	
	116.1834		21.1834
100.0000		10.1239	
	86.0708		0.0000

The risk-neutral probability of an upward movement is:

$$p^* = \frac{e^{(r-\delta)h} - d}{u-d} = \frac{e^{(0.08-0.05)(0.25)} - 0.86071}{1.16183 - 0.86071} = 0.48757$$

The value of the call option is:

$$V = e^{-rh}\left[(p^*)V_u + (1-p^*)V_d\right] = e^{-0.08(0.25)}\left[0.48757(21.1834) + (1-0.48757)(0.00)\right]$$
$$= 10.1239$$

The realistic probability can now be determined:

$$V = e^{-\gamma h}\left[(p)V_u + (1-p)V_d\right]$$
$$10.1239 = e^{-(0.2153)(0.25)}\left[(p)(21.1834) + (1-p)(0.0000)\right]$$
$$p = \frac{10.1239e^{(0.2153)(0.25)}}{21.1834}$$
$$p = 0.50434$$

Now that we have the realistic probability, we can find the expected return for the stock:

$$S = e^{-\alpha h}\left[(p)Sue^{\delta h} + (1-p)Sde^{\delta h}\right]$$
$$100 = e^{-\alpha(0.25)}\left[(0.50434)(116.1834)e^{0.05(0.25)} + (1-0.50434)(86.0708)e^{0.05(0.25)}\right]$$
$$\alpha = \ln\left(\frac{(0.50434)(116.1834)e^{0.05(0.25)} + (1-0.50434)(86.0708)e^{0.05(0.25)}}{100}\right)\times\frac{1}{0.25}$$
$$= 0.100$$

Solution 23

D Chapter 11, Jarrow and Rudd Binomial Tree

The values of u and d are:

$$u = e^{(r-\delta-0.5\sigma^2)h+\sigma\sqrt{h}} = e^{(0.07-0.00-0.5\times0.30^2)(1)+0.3\sqrt{1}} = 1.38403$$
$$d = e^{(r-\delta-0.5\sigma^2)h-\sigma\sqrt{h}} = e^{(0.07-0.00-0.5\times0.30^2)(1)-0.3\sqrt{1}} = 0.75957$$

The possible stock prices at the end of one year are:

$$38u = 38\times1.38403 = 52.5932$$
$$38d = 38\times0.75957 = 28.8637$$

The payoffs for the call option in 1 year are:

Up state: $Max[0,\ 52.5932-40] = 12.5932$

Down state: $Max[0,\ 28.8637-40] = 0.0000$

The stock price tree and the call price tree are below:

Stock		Call	
	52.5932		12.5932
38.0000		5.8842	
	28.8637		0.0000

The risk-neutral probability of an upward movement is:

$$p^* = \frac{e^{(r-\delta)h} - d}{u-d} = \frac{e^{(0.07-0.00)(1)} - 0.75957}{1.38403 - 0.75957} = 0.50113$$

The value of the call option is:

$$V = e^{-rh}\left[(p^*)V_u + (1-p^*)V_d\right] = e^{-0.07(1)}\left[0.50113(12.5932) + (1-0.50113)(0.0000)\right]$$
$$= 5.884$$

Solution 24

C Chapter 11, Jarrow and Rudd Binomial Tree

The values of u and d are:

$$u = e^{(r-\delta-0.5\sigma^2)h+\sigma\sqrt{h}} = e^{(0.07-0.00-0.5\times 0.30^2)(1)+0.3\sqrt{1}} = 1.38403$$
$$d = e^{(r-\delta-0.5\sigma^2)h-\sigma\sqrt{h}} = e^{(0.07-0.00-0.5\times 0.30^2)(1)-0.3\sqrt{1}} = 0.75957$$

The possible stock prices at the end of one year are:

$$38u = 38 \times 1.38403 = 52.5932$$
$$38d = 38 \times 0.75957 = 28.8637$$

The stock price tree and the put price tree are below:

Stock		Put	
	52.5932		0.0000
38.0000		5.1799	
	28.8637		11.1363

The risk-neutral probability of an upward movement is:

$$p^* = \frac{e^{(r-\delta)h} - d}{u-d} = \frac{e^{(0.07-0.00)(1)} - 0.75957}{1.38403 - 0.75957} = 0.50113$$

The value of the put option is:

$$V = e^{-rh}\left[(p^*)V_u + (1-p^*)V_d\right] = e^{-0.07(1)}\left[0.50113(0.0000) + (1-0.50113)(11.1363)\right]$$
$$= 5.180$$

Solution 25

C Chapter 11, Jarrow and Rudd Binomial Tree

The values of u and d are:

$$u = e^{(r-\delta-0.5\sigma^2)h+\sigma\sqrt{h}} = e^{(0.08-0.05-0.5\times0.30^2)(0.25)+0.3\sqrt{0.25}} = 1.15749$$
$$d = e^{(r-\delta-0.5\sigma^2)h-\sigma\sqrt{h}} = e^{(0.08-0.05-0.5\times0.30^2)(0.25)-0.3\sqrt{0.25}} = 0.85749$$

The stock price tree and the call price tree are below:

Stock		Call	
	115.7486		20.7486
100.0000		10.1717	
	85.7486		0.0000

The risk-neutral probability of an upward movement is:

$$p^* = \frac{e^{(r-\delta)h} - d}{u-d} = \frac{e^{(0.08-0.05)(0.25)} - 0.85749}{1.15749 - 0.85749} = 0.50014$$

The value of the call option is:

$$V = e^{-rh}\left[(p^*)V_u + (1-p^*)V_d\right] = e^{-0.08(0.25)}\left[0.50014(20.7486) + (1-0.50014)(0.0000)\right]$$
$$= 10.172$$

Solution 26

D Chapter 11, Jarrow and Rudd Binomial Tree

The values of u and d are:

$$u = e^{(r-\delta-0.5\sigma^2)h+\sigma\sqrt{h}} = e^{(0.08-0.05-0.5\times0.30^2)(0.25)+0.3\sqrt{0.25}} = 1.15749$$
$$d = e^{(r-\delta-0.5\sigma^2)h-\sigma\sqrt{h}} = e^{(0.08-0.05-0.5\times0.30^2)(0.25)-0.3\sqrt{0.25}} = 0.85749$$

The stock price tree and the call price tree are below:

Stock		Call	
	115.7486		20.7486
100.0000		10.1717	
	85.7486		0.0000

The risk-neutral probability of an upward movement is:

$$p^* = \frac{e^{(r-\delta)h} - d}{u-d} = \frac{e^{(0.08-0.05)(0.25)} - 0.85749}{1.15749 - 0.85749} = 0.50014$$

The value of the call option is:

$$V = e^{-rh}\left[(p^*)V_u + (1-p^*)V_d\right] = e^{-0.07(1)}\left[0.50014(20.7486) + (1-0.50014)(0.0000)\right]$$
$$= 10.1717$$

The realistic probability can now be determined:

$$V = e^{-\gamma h}\left[(p)V_u + (1-p)V_d\right]$$
$$10.1717 = e^{-(0.5281)(0.25)}\left[(p)(20.7486) + (1-p)(0.0000)\right]$$
$$p = \frac{10.1717e^{(0.5281)(0.25)}}{20.7486}$$
$$p = 0.55943$$

Now that we have the realistic probability, we can find the expected return for the stock:

$$p = \frac{e^{(\alpha-\delta)h} - d}{u-d}$$
$$0.55943 = \frac{e^{(\alpha-0.05)\times 0.25} - 0.85749}{1.15749 - 0.85749}$$
$$\alpha = 0.150$$

Solution 27

A Chapter 11, Jarrow and Rudd Binomial Tree

The values of u and d are:

$$u = e^{(r-\delta-0.5\sigma^2)h+\sigma\sqrt{h}} = e^{(0.12-0.045-0.5\times 0.27^2)(0.5)+0.27\sqrt{0.5}} = 1.23392$$
$$d = e^{(r-\delta-0.5\sigma^2)h-\sigma\sqrt{h}} = e^{(0.12-0.045-0.5\times 0.27^2)(0.5)-0.27\sqrt{0.5}} = 0.84228$$

The possible stock prices at the end of one year are:

$$38u = 53\times 1.23392 = 65.3976$$
$$38d = 53\times 0.84228 = 44.6408$$

The stock price tree and the call price tree are below:

Stock		Call	
	65.3976		5.3976
53.0000		2.5431	
	44.6408		0.0000

The value of delta is:

$$\Delta = e^{-\delta h}\frac{V_u - V_d}{S(u-d)} = e^{-0.045(0.5)}\frac{5.3976 - 0.0000}{65.3976 - 44.6408} = 0.254$$

Solution 28

A Chapter 11, Jarrow and Rudd Binomial Tree

The values of u and d are:

$$u = e^{(r-\delta-0.5\sigma^2)h+\sigma\sqrt{h}} = e^{(0.05-0.01-0.5\times0.35^2)(0.25)+0.35\sqrt{0.25}} = 1.18493$$
$$d = e^{(r-\delta-0.5\sigma^2)h-\sigma\sqrt{h}} = e^{(0.05-0.01-0.5\times0.35^2)(0.25)-0.35\sqrt{0.25}} = 0.83501$$

The stock price tree, the put payoffs and the risk-neutral probability of each payoff are below:

					Put Payoff	# of Paths	Probability
Stock				98.5706	0.0000	1	0.06261
			83.1865				
		70.2035		69.4615	0.0000	4	0.25022
	59.2467		58.6206				
50.0000		49.4716		48.9487	0.0000	6	0.37500
	41.7505		41.3092				
		34.8620		34.4936	0.0000	4	0.24978
			29.1101				
				24.3072	9.6928	1	0.06239

Although we show the entire table above, there is no need to calculate the stock prices higher than $34.4936 to answer this question. At those prices, the option is clearly out-of-the-money.

Only one of the final stock prices is below the strike price of $34. The payoff if the lowest stock price is reached is:

$$34 - 24.3072 = 9.6928$$

The risk-neutral probability of an upward movement is:

$$p^* = \frac{e^{(r-\delta)h} - d}{u-d} = \frac{e^{(0.05-0.01)(0.25)} - 0.83501}{1.18493 - 0.83501} = 0.50022$$

The probability of 4 downward movements is:

$$(1-p^*)^4 = (1-0.50022)^4 = 0.06239$$

Since the option is a European option, we can find its value directly as the present value of its expected value:

$$V(S_0, K, 0) = e^{-r(hn)} \sum_{i=0}^{n} \left[\binom{n}{i} (p^*)^{(n-i)} (1-p^*)^i V(S_0 u^{n-i} d^i, K, hn) \right]$$
$$= e^{-0.05} \times 0.06239 \times 9.6928 = 0.5752$$

Solution 29

D Chapter 11, J-R and CRR Binomial Trees

In the J-R model, the values of u and d are:

$$u = e^{(r-\delta-0.5\sigma^2)h+\sigma\sqrt{h}} = e^{(0.07-0.025-0.5\times0.30^2)(0.25)+0.30\sqrt{0.25}} = e^{0+0.30\sqrt{0.25}}$$
$$d = e^{(r-\delta-0.5\sigma^2)h-\sigma\sqrt{h}} = e^{(0.07-0.025-0.5\times0.30^2)(0.25)-0.30\sqrt{0.25}} = e^{0-0.30\sqrt{0.25}}$$

In the CRR model, the values of u and d are:

$$u = e^{\sigma\sqrt{h}} = e^{0.30\sqrt{0.25}}$$
$$d = e^{-\sigma\sqrt{h}} = e^{-0.30\sqrt{0.25}}$$

Since this J-R model's up and down factors are the same as this CRR model's up and down factors, Ann and Betsy are using the same binomial model. Therefore, the prices they produce satisfy put-call parity:

$$C_{Eur} + Ke^{-rT} = S_0e^{-\delta T} + P_{Eur}$$
$$A + 50e^{-0.07\times1} = 50e^{-0.025\times1} + B$$
$$A - B = 2.1458$$

Solution 30

B Chapter 11, Alternative Binomial Trees

If the option has a positive payoff in both the up and down states, then:

$$\Delta = e^{-\delta h}\frac{V_u - V_d}{S(u-d)} = e^{-0.11\times0.25}\frac{(38-Su)-(38-Sd)}{Su-Sd} = 0.9729\frac{Sd-Su}{Su-Sd} = -0.9729$$

But $\Delta = -0.262$, so it must be the case that $V_u = 0$.

Since we are given the values of Δ and B, we can solve for u and d:

$$-0.262 = \Delta = e^{-\delta h}\frac{V_u - V_d}{S(u-d)} = e^{-0.11\times0.25}\frac{0-(38-40d)}{40u-40d} = 0.9729\frac{40d-38}{40(u-d)}$$
$$11.76 = B = e^{-rh}\frac{uV_d - dV_u}{u-d} = e^{-0.15\times0.25}\frac{u(38-40d)-d\times0}{u-d} = 0.9632\frac{u(38-40d)}{u-d}$$

Dividing the first equation by the second allows us to solve for u:

$$\frac{-0.262}{11.76} = \frac{0.9729\dfrac{40d-38}{40(u-d)}}{0.9632\dfrac{u(38-40d)}{u-d}}$$
$$-0.02228 = 0.02525\times\frac{-1}{u}$$
$$u = 1.1334$$

Now we can find the value of d:

$$-0.262 = 0.9729\frac{40d - 38}{40(u - d)}$$
$$-0.262 = 0.9729\frac{40d - 38}{40(1.1334 - d)}$$
$$-12.2094 + 10.7722d = 40d - 38$$
$$d = 0.8824$$

Both the Cox-Ross-Rubinestein model and the Jarrow-Rudd model have the same ratio of u to d:

$$\frac{u}{d} = e^{2\sigma\sqrt{h}}$$
$$\frac{1.1334}{0.8824} = e^{2\sigma\sqrt{0.25}}$$
$$\sigma = 0.2503$$

Solution 31

C Chapter 11, Utility Values and State Prices

The payoffs of the call option are:

$$V_u = Max(0, 200 - 130) = 70$$
$$V_d = Max(0, 50 - 130) = 0$$

The price of the call option is:

$$V = Q_u V_u + Q_d V_d = pU_u \times 70 + (1-p)U_d \times 0 = 0.55(0.9) \times 70 + 0.45(1.03) \times 0 = 34.65$$

Solution 32

E Chapter 11, Utility Values and State Prices

The payoffs of the call option are:

$$V_u = Max(0, 200 - 130) = 70$$
$$V_d = Max(0, 50 - 130) = 0$$

The price of the call option is:

$$V = Q_u V_u + Q_d V_d = pU_u \times 70 + (1-p)U_d \times 0 = 0.55(0.9) \times 70 + 0.45(1.03) \times 0 = 34.65$$

The expected return on the call option can now be calculated:

$$(1 + \gamma_{Call})^{0.5} = \frac{pV_u + (1-p)V_d}{V}$$
$$(1 + \gamma_{Call})^{0.5} = \frac{0.55 \times 70 + 0.45 \times 0}{34.65}$$
$$\gamma_{Call} = 0.2346$$

The payoffs of the put option are:

$$V_u = Max(0, 100 - 200) = 0$$
$$V_d = Max(0, 100 - 50) = 50$$

The price of the put option is:

$$V = Q_u V_u + Q_d V_d = pU_u \times 0 + (1-p)U_d \times 50 = 0.55(0.9) \times 0 + 0.45(1.03) \times 50$$
$$= 23.175$$

The expected return on the put option can now be calculated:

$$(1+\gamma_{Put})^{0.5} = \frac{pV_u + (1-p)V_d}{V}$$
$$(1+\gamma_{Put})^{0.5} = \frac{0.55 \times 0 + 0.45 \times 50}{23.175}$$
$$\gamma_{Put} = -0.0574$$

The difference in the expected returns is:

$$\gamma_{Call} - \gamma_{Put} = 0.2346 - (-0.0574) = 0.2920$$

Solution 33

E Chapter 11, Utility Values and State Prices

The price of the stock is:

$$S = Q_u S_u e^{\delta h} + Q_d S_d e^{\delta h} = pU_u S_u e^{\delta h} + (1-p)U_d S_d e^{\delta h}$$
$$= 0.55 \times 0.87 \times 200e^{0.06\times 1} + 0.45 \times 0.98 \times 50e^{0.06\times 1} = 125.0313$$

The expected return can now be calculated:

$$(1+\alpha)^h = \frac{pS_u e^{\delta h} + (1-p)S_d e^{\delta h}}{S}$$
$$(1+\alpha)^1 = \frac{0.55 \times 200e^{0.06\times 1} + 0.45 \times 50e^{0.06\times 1}}{125.0313}$$
$$\alpha = 0.1253$$

Solution 34

D Chapter 11, Utility Values and State Prices

Although we called the two states “up” and “down” in the Review Notes, any consistent labeling system is valid. In the formulas, we can replace references to the up state with references to Scenario 1, and we can replace references to the down state with references to Scenario 2.

The price of the stock is:

$$S = e^{-\alpha h}\left[pS_1e^{\delta h} + (1-p)S_2e^{\delta h}\right] = e^{-0.07\times 1}\left[0.3\times 50e^{0\times 1} + 0.7\times 100e^{0\times 1}\right]$$
$$= 79.2535$$

We can use Stock X and the risk-free asset to obtain two equations and two unknowns:

$$79.2535 = 50Q_1 + 100Q_2$$
$$0.92 = Q_1 + Q_2$$

Multiplying the bottom equation by 50 and subtracting it from the top equation, we have:

$$79.2535 - 50\times 0.92 = 50Q_1 - 50Q_1 + 100Q_2 - 50Q_2$$
$$33.2535 = 50Q_2$$
$$Q_2 = 0.6651$$

Solution 35

A Chapter 11, Utility Values and State Prices

Although the textbook presented state prices with 2 states, it is not difficult to extend the analysis to cover 3 states.

We can find the utility value for the third state:

$$S = Q_1S_1e^{\delta h} + Q_2S_2e^{\delta h} + Q_3S_3e^{\delta h}$$
$$S = p_1U_1S_1e^{\delta h} + p_2U_2S_2e^{\delta h} + p_3U_3S_3e^{\delta h}$$
$$100 = 0.46\times 0.8554\times 200 + 0.06\times U_2\times 120 + 0.48\times 0.9819\times 35$$
$$U_2 = 0.6677$$

The values of the call option on Stock B at time T are:

$$V_1 = Max(0, 70-60) = 10$$
$$V_2 = Max(0, 90-60) = 30$$
$$V_3 = Max(0, 40-60) = 0$$

The value of the option is:

$$V = Q_1V_1 + Q_2V_2 + Q_3V_3$$
$$V = p_1U_1V_1 + p_2U_2V_2 + p_3U_3V_3$$
$$V = 0.46\times 0.8554\times 10 + 0.06\times 0.6677\times 30 + 0.48\times 0.9819\times 0 = 5.137$$

Solution 36

D Chapter 11, Utility Values and State Prices

This stock has a higher value in the low state than it does in the high state. Although this situation does not affect our solution to the question, it suggests that the stock could be used as a hedge for whatever asset (which might, for example, be the entire market) is used to define high state and low state.

The price of the risk-free asset is:

$$110e^{-rh} = 110(Q_u + Q_d) = 110\left(pU_u + (1-p)U_d\right) = 110\left(0.4\times 0.85 + 0.6\times 0.95\right)$$
$$= 100.10$$

The price of the stock is:

$$S = Q_u S_u e^{\delta h} + Q_d S_d e^{\delta h} = pU_u S_u e^{\delta h} + (1-p)U_d S_d e^{\delta h}$$
$$= 0.40\times 0.85\times 75e^{0.05\times 1} + 0.60\times 0.95\times 125e^{0.05\times 1} = 101.71$$

The stock price exceeds the price of the risk-free bond by:

$$101.71 - 100.10 = 1.61$$

Solution 37

A Chapter 11, Utility Values and State Prices

Since the probability of the high state is 0.65, the probability of the low state is:

$$1 - p = 1 - 0.65 = 0.35$$

The stock's cash flow in the low state can now be determined:

$$S = Q_u S_u e^{\delta h} + Q_d S_d e^{\delta h}$$
$$S = pU_u S_u e^{\delta h} + (1-p)U_d S_d e^{\delta h}$$
$$36.53 = 0.65\times 0.90\times 50e^{0\times 2} + 0.35\times 1.04\times C_L e^{0\times 2}$$
$$C_L = 20$$

The payoffs of the derivative are:

$$V_u = \ln(50^3) = 11.7361$$
$$V_d = \ln(20^3) = 8.9872$$

The value of the derivative is:

$$V = Q_u V_u + Q_d V_d = pU_u \times V_u + (1-p)U_d \times V_d$$
$$= 0.65(0.9)\times 11.7361 + 0.35(1.04)\times 8.9872 = 10.1369$$

Solution 38

C Chapter 11, Utility Values and State Prices

Since the probability of the high state is 0.65, the probability of the low state is:

$$1 - p = 1 - 0.65 = 0.35$$

The stock's cash flow in the low state can now be determined:

$$S = Q_u S_u e^{\delta h} + Q_d S_d e^{\delta h}$$
$$S = pU_u S_u e^{\delta h} + (1-p)U_d S_d e^{\delta h}$$
$$36.53 = 0.65 \times 0.90 \times 50e^{0\times 2} + 0.35 \times 1.04 \times C_L e^{0\times 2}$$
$$C_L = 20$$

The risk-free rate can be found with the probabilities and utility values:

$$e^{-rh} = Q_u + Q_d$$
$$e^{-rh} = pU_u + (1-p)U_d$$
$$e^{-r\times 2} = 0.65 \times 0.90 + 0.35 * 1.04$$
$$r = 0.0262$$

The payoffs of the derivative at time 3 are:

$$V_u(3) = \ln(50^3) = 11.7361$$
$$V_d(3) = \ln(20^3) = 8.9872$$

Although the derivative does not pay until time 3, its payoffs are known with certainty at time 2. Therefore, we can discount the payoffs back to time 2 at the risk-free interest rate.

$$V_u(2) = V_u(3)e^{-r\times 1} = 11.7361e^{-0.0262} = 11.4329$$
$$V_d(2) = V_d(3)e^{-r\times 1} = 8.9872e^{-0.0262} = 8.7550$$

The value of the derivative is:

$$V(0) = Q_u V_u(2) + Q_d V_d(2) = pU_u \times V_u(2) + (1-p)U_d \times V_d(2)$$
$$= 0.65 \times 0.9 \times 11.4329 + 0.35 \times 1.04 \times 8.7550 = 9.8751$$

Solution 39

B Chapter 11, Utility Values and State Prices

The expected return on the call options can be found with the formula below:

$$1 + \gamma_{Call} = \frac{pV_u + (1-p)V_d}{V} = \frac{pV_u + (1-p)V_d}{pU_u V_u + (1-p)U_d V_d}$$

The expected returns for the first 2 choices are:

A: $\gamma_{Call} = \frac{pV_u + (1-p)V_d}{pU_uV_u + (1-p)U_dV_d} - 1 = \frac{0.6 \times 60 + 0.4 \times 20}{0.6 \times 0.92 \times 60 + 0.4 \times 1.05 \times 20} - 1 = 5.97\%$

B: $\gamma_{Call} = \frac{pV_u + (1-p)V_d}{pU_uV_u + (1-p)U_dV_d} - 1 = \frac{0.6 \times 35 + 0.4 \times 0}{0.6 \times 0.92 \times 35 + 0.4 \times 1.05 \times 0} - 1 = 8.70\%$

For Choice C, the strike price is greater than either of the possible stock prices, so the value of the call option is zero. Consequently, the expected return is undefined for the option described in Choice C.

The expected returns for the fourth and fifth choices are:

D: $\gamma_{Put} = \frac{pV_u + (1-p)V_d}{pU_uV_u + (1-p)U_dV_d} - 1 = \frac{0.6 \times 0 + 0.4 \times 15}{0.6 \times 0.92 \times 0 + 0.4 \times 1.05 \times 15} - 1 = -4.76\%$

E: $\gamma_{Put} = \frac{pV_u + (1-p)V_d}{pU_uV_u + (1-p)U_dV_d} - 1 = \frac{0.6 \times 20 + 0.4 \times 60}{0.6 \times 0.92 \times 20 + 0.4 \times 1.05 \times 60} - 1 = -0.66\%$

The option described in Choice B has the highest expected return.

Solution 40

B Chapter 11, Arbitrage

The Cox-Ross-Rubinstein model is:

$$u = e^{\sigma\sqrt{h}}$$

$$d = e^{-\sigma\sqrt{h}}$$

The Jarrow-Rudd model is:

$$u = e^{(r-\delta-0.5\sigma^2)h+\sigma\sqrt{h}}$$

$$d = e^{(r-\delta-0.5\sigma^2)h-\sigma\sqrt{h}}$$

From Chapter 10, we know that to avoid arbitrage, a model must satisfy the following:

$$d < e^{(r-\delta)h} < u$$

The values of d, $e^{(r-\delta)h}$, and u are calculated for each model below:

Model	r	δ	σ	h	d	$e^{(r-\delta)h}$	u	Type
A	0.25	0.01	0.30	0.25	0.8607	1.0618	1.1618	CRR
B	0.21	0.00	0.20	1.00	0.8187	**1.2337**	**1.2214**	CRR
C	0.14	0.03	0.10	0.50	0.9317	1.0565	1.0733	CRR
D	0.14	0.05	0.50	0.50	0.6900	1.0460	1.3994	J-R
E	0.07	0.00	0.10	1.00	0.9656	1.0725	1.1794	J-R

Model B violates the restriction because it has the risk-free asset earning more than the risky asset after an upward move:

$$1.2337 > 1.2214$$

Solution 41

B Chapter 11, Realistic Probability

We can solve for p, the true probability of the stock price going up, using the following formula:

$$p = \frac{e^{(\alpha-\delta)h} - d}{u-d} = \frac{e^{(0.10-0)\times 1} - 0.763}{1.477 - 0.763} = 0.47923$$

Solution 42

D Chapter 11, Greeks in the Jarrow-Rudd Binomial Model

How do we know that gamma in the question refers to Γ and not γ? Because we would need to know the realistic probability of an upward movement in order to determine the expected return on the option, γ, but there is no way of knowing the realistic probability of an upward movement.

The values of u and d are:

$$u = e^{(r-\delta-0.5\sigma^2)h+\sigma\sqrt{h}} = e^{(0.11-0.04-0.5\times 0.32^2)(1)+0.32\sqrt{1}} = 1.40326$$
$$d = e^{(r-\delta-0.5\sigma^2)h-\sigma\sqrt{h}} = e^{(0.11-0.04-0.5\times 0.32^2)(1)-0.32\sqrt{1}} = 0.73993$$

The risk-neutral probability of an upward movement is:

$$p^* = \frac{e^{(r-\delta)h} - d}{u-d} = \frac{e^{(0.11-0.04)(1)} - 0.73993}{1.40326 - 0.73993} = 0.50137$$

The stock price tree and its corresponding tree of option prices are:

Stock			American Put		
		78.7658			0.0000
	56.1305			0.2088	
40.0000		41.5326	5.6339		0.4674
	29.5972			**12.4028**	
		21.8998			20.1002

If the stock price initially moves down, then the resulting put price is \$12.4028. This price is in bold type above to indicate that it is optimal to exercise early at this node:

$$42 - 29.5972 = 12.4028$$

This exercise value of 12.4028 is greater than the value of holding the option, which is:

$$e^{-0.11(1)}\left[(0.50137)(0.4674) + (1-0.50137)(20.1002)\right] = 9.1884$$

There is no need to calculate the current value of the American put to answer this question, but it is provided in the tree above for completeness.

We need to calculate the two possible values of delta at time 1:

$$\Delta(Su,h) = e^{-\delta h}\frac{V_{uu} - V_{ud}}{Su^2 - Sud} = e^{-0.04\times 1}\frac{0.0000 - 0.4674}{78.7658 - 41.5326} = -0.0121$$

$$\Delta(Sd,h) = e^{-\delta h}\frac{V_{ud} - V_{dd}}{Sud - Sd^2} = e^{-0.04\times 1}\frac{0.4674 - 20.1002}{41.5326 - 21.8998} = -0.9608$$

We can now calculate gamma:

$$\Gamma(S,0) \approx \Gamma(S_h,h) = \frac{\Delta(Su,h) - \Delta(Sd,h)}{Su - Sd} = \frac{-0.0121 - (-0.9608)}{56.1305 - 29.5972} = 0.0358$$

Solution 43

D Chapter 11, State Prices

Since the strike price is \$10, the put option pays Max[0, 10 – 15] = \$0 if the up state occurs. If the down state occurs, the call option pays Max[0, 10 – 7] = \$3. We can use the stock price and the option price to solve for the original state prices:

$$\left.\begin{aligned} 10 &= 15Q_u + 7Q_d \\ 1.69 &= 0Q_u + 3Q_d \end{aligned}\right\} \Rightarrow \quad \begin{aligned} Q_u &= 0.4038 \\ Q_d &= 0.5633 \end{aligned}$$

The discount factor for one year is:

$$e^{-r} = Q_u + Q_d = 0.4038 + 0.5633 = 0.9671$$

After the correction is made, the new state prices still result in the same risk-free rate of return:

$$e^{-r} = \hat{Q}_u + \hat{Q}_d = 0.9671$$

We can use the stock price with the corrected value after a down-move and the expression for the risk-free discount factor to obtain another system of 2 equations and 2 unknowns:

$$\left.\begin{aligned} 10 &= 15\hat{Q}_u + 5\hat{Q}_d \\ 0.9671 &= \hat{Q}_u + \hat{Q}_d \end{aligned}\right\} \Rightarrow \quad \hat{Q}_d = 0.4507$$

The new state price can now be used to find the value of the option. Since the payoff of the put option is Max[0, 10 – 5] = \$5 when we use the correct lower stock price of \$5, we have the following price for the put option:

$$0\hat{Q}_u + 5\hat{Q}_d = 5\times 0.4507 = 2.2533$$

<u>Alternate Solution</u>

We don't have to use state prices to answer this question.

We can use the stock price and the option price to solve for the risk-neutral probability of an upward movement and the discount factor:

$$\left.\begin{array}{l} 10 = e^{-r}\left[15p^* + 7(1-p^*)\right] \\ 1.69 = e^{-r}\left[0p^* + 3(1-p^*)\right] \end{array}\right\} \Rightarrow \begin{array}{l} e^{-r} = 0.9671 \\ p^* = 0.4175 \end{array}$$

After the correction is made, we still have the same discount factor of 0.9671, but we have a new risk-neutral probability of an upward movement. We can use the stock price with the corrected value after a down-move to solve for the new, corrected risk-neutral probability of an up move:

$$10 = 0.9671\left[15\hat{p}^* + 5(1-\hat{p}^*)\right]$$
$$\frac{10}{0.9671} = 10\hat{p}^* + 5$$
$$\hat{p}^* = 0.5340$$

The payoff of the put option is \$5 when we use the correct lower stock price of \$5. We have the following price for the put option:

$$5(1-\hat{p}^*)e^{-r} = 5(1-0.5340)(0.9671) = 2.2533$$

Solution 44

E Chapter 11, Utility Values and State Prices

The payoffs of the call option are:

$$V_u = Max(0, 100 - K_C) = 100 - K_C$$
$$V_d = Max(0, 60 - K_C) = 0$$

The price of the call option is:

$$V = Q_u V_u + Q_d V_d = pU_u \times (100 - K_C) + (1-p)U_d \times 0 = 0.6(0.92)(100 - K_C)$$

The expected return on the call option can now be calculated:

$$1 + \gamma_{Call} = \frac{pV_u + (1-p)V_d}{V}$$
$$1 + \gamma_{Call} = \frac{0.6(100 - K_C)}{0.6(0.92)(100 - K_C)}$$
$$1 + \gamma_{Call} = \frac{1}{0.92}$$
$$\gamma_{Call} = 0.0870$$

The payoffs of the put option are:

$$V_u = Max(0, K_P - 100) = 0$$
$$V_d = Max(0, K_P - 60) = K_P - 60$$

The price of the put option is:

$$V = Q_u V_u + Q_d V_d = pU_u \times 0 + (1-p)U_d \times (K_P - 60) = 0.4(1.05)(K_P - 60)$$

The expected return on the put option can now be calculated:

$$1+\gamma_{Put} = \frac{pV_u + (1-p)V_d}{V}$$

$$1+\gamma_{Put} = \frac{0.4(K_P - 60)}{0.4(1.05)(K_P - 60)}$$

$$1+\gamma_{Put} = \frac{1}{1.05}$$

$$\gamma_{Put} = -0.0476$$

The difference in the expected returns is:

$$\gamma_{Call} - \gamma_{Put} = 0.0870 - (-0.0476) = 0.1346$$

Solution 45

B Chapter 11, Utility Values and State Prices

Notice that the expected return on the call option is expressed as an annual effective rate, not a continuously compounded rate.

Since the call option consists of 20 of the up-state Arrow-Debreu securities, the expected return of each up-state Arrow-Debreu security is equal to the expected return of the call option:

$$\gamma_u = \gamma_{Call} = 0.23$$

The utility value is the present value of \$1, calculated using the expected return of the corresponding Arrow-Debreu security. Therefore, the up-state utility value is:

$$U_u = \frac{1}{1+\gamma_u} = \frac{1}{1.23}$$

The call option pays off \$20 in the up state and \$0 in the down state. We can use the call option to obtain the state price for the up state:

$$7.37 = 20Q_u + 0Q_d$$

$$Q_u = \frac{7.37}{20} = 0.3685$$

We can use the state price of the up-state to obtain the realistic probability that the up state occurs:

$$Q_u = p \times U_u$$

$$0.3685 = p \times \frac{1}{1.23}$$

$$p = 0.4533$$

We can now find the down-state utility value:

$$S = Q_u S_u + Q_d S_d$$
$$S = U_u p S_u + U_d (1-p) S_d$$
$$76 = \frac{1}{1.23} \times 0.4533 \times 100 + U_d (1-0.4533)70$$
$$U_d = 1.0229$$

The sum of the up-state and the down-state utility values is:

$$U_u + U_d = \frac{1}{1.23} + 1.0229 = 1.8359$$

