March 27, 2008 Update: This excerpt from the Chapter 24 Review Note addresses the changes resulting from the recent errata update regarding pages 780 through 788 of the Derivatives Market textbook. Please see the ActuarialBrew.com errata for more details.

Where:

$$P_r = \frac{\partial P}{\partial r} = Delta$$
$$P_{rr} = \frac{\partial^2 P}{\partial r^2} = Gamma$$
$$P_t = \frac{\partial P}{\partial t}$$

Delta and gamma are defined as the first and second partial derivatives with respect to the short-term interest rate.

This is similar to the definition of delta and gamma for stocks that we saw in the Chapter 12 Review Note, but for stocks, delta and gamma are the first and second partial derivatives with respect to the stock price.

Delta and Gamma for Bonds

The delta of a bond is the partial derivative of its price with respect to the short-term interest rate:

$$Delta = P_r = \frac{\partial P}{\partial r}$$

The gamma of a bond is the second partial derivative of its price with respect to the short-term interest rate:

$$Gamma = P_{rr} = \frac{\partial^2 P}{\partial r^2}$$

Let's define two more functions:

$$\alpha(r,t,T) = \frac{1}{P}\left[a(r)P_r + \frac{1}{2}[\sigma(r)]^2 P_{rr} + P_t\right]$$
$$q(r,t,T) = -\frac{1}{P}\sigma(r)P_r$$

Now we can describe the change in the price of P with an Itô process containing $\alpha(r,t,T)$ and $q(r,t,T)$:

$$\frac{dP}{P} = \frac{1}{P}\left[a(r)P_r + \frac{1}{2}[\sigma(r)]^2 P_{rr} + P_t\right]dt + \frac{1}{P}\sigma(r)P_r dZ$$
$$\frac{dP}{P} = \alpha(r,t,T)dt - q(r,t,T)dZ$$

Itô Process for Zero-Coupon Bond Prices

At time t, the price of a zero-coupon bond that matures at time T follows an Itô process:

$$\frac{dP}{P} = \alpha(r,t,T)dt - q(r,t,T)dZ$$

where:

$$\alpha(r,t,T) = \frac{1}{P}\left[a(r)P_r + \frac{1}{2}[\sigma(r)]^2 P_{rr} + P_t\right]$$

$$q(r,t,T) = -\frac{1}{P}\sigma(r)P_r$$

The instantaneous rate of return on a bond maturing at time T has a mean of $\alpha(r,t,T)$ and a standard deviation of $q(r,t,T)$. That is, the expected return on the bond over the next instant (expressed as an annual return) is $\alpha(r,t,T)$.

Although a wide range of parameters are possible, it is usually the case that $\alpha(r,t,T) > r$. The bond is expected to earn more than the risk-free rate because owning a bond is riskier than lending at the risk-free rate. Since $\sigma(r)$ is positive, $q(r,t,T)$ is positive, because $P_r < 0$.

Let's consider two bonds, one that matures at time T_1 and one that matures at time T_2. Their Itô processes are:

$$\frac{dP(r,t,T_1)}{P(r,t,T_1)} = \alpha(r,t,T_1)dt - q(r,t,T_1)dZ$$

$$\frac{dP(r,t,T_2)}{P(r,t,T_2)} = \alpha(r,t,T_2)dt - q(r,t,T_2)dZ$$

This is similar to the situation we encountered in Section 20.4 of Chapter 20, when we considered two assets that had the same source of randomness. Using the same kind of analysis as in Section 20.4, we will show that the Sharpe ratios of the two bonds must be equal:

$$\frac{\alpha(r,t,T_1) - r}{q(r,t,T_1)} = \frac{\alpha(r,t,T_2) - r}{q(r,t,T_2)}$$

If at time t, we buy a zero-coupon bond maturing at time T_2 and buy N zero-coupon bonds that mature at time T_1, then the delta of the resulting portfolio can be found using the formula from the Chapter 12 Review Note for the delta of a portfolio:

$$Delta_{Port} = \sum_{i=1}^{n} q_i Delta_i = P_r(r,t,T_2) + N \times P_r(r,t,T_1)$$

We can solve for the number of the T_1-year bonds to buy in order to create a portfolio with a delta of zero, thereby delta-hedging the purchase of the T_2-year bond:

$$P_r(r,t,T_2) + N \times P_r(r,t,T_1) = 0$$
$$N = -\frac{P_r(r,t,T_2)}{P_r(r,t,T_1)}$$

If N is negative, then we sell rather than buy the T_2-year bond.

As in the Chapter 13 Review Note, when creating a delta-hedged portfolio, we lend any excess funds created by the position and borrow any shortfall. The amount of funds lent at the short-term rate is W:

$$W = \text{Lending} = -P(r,t,T_2) - N \times P(r,t,T_1)$$

If W is positive, then we lend. If W is negative, then we borrow.

The lending does not affect the delta of the portfolio, because the funds are lent at the short-term rate, so the lending has a delta of zero.

Delta-Hedging

The purchase of a T_2-year zero-coupon bond can be delta-hedged with the following 2 steps:

1. Purchase N T_1-year zero-coupon bonds:

$$N = -\frac{P_r(r,t,T_2)}{P_r(r,t,T_1)}$$

2. Lend W:

$$W = -P(r,t,T_2) - N \times P(r,t,T_1)$$

Let's use $I(t)$ to denote the value of the delta-hedged position. The value of the delta-hedged position at time t is:

$$I(t) = NP(r,t,T_1) + P(r,t,T_2) + W(t) = 0$$

The change in $I(t)$ is:

$$\begin{aligned} dI(t) &= N \times dP(r,t,T_1) + dP(r,t,T_2) + rW(t)dt \\ &= N\left[\alpha(r,t,T_1)dt - q(r,t,T_1)dZ\right]P(r,t,T_1) \\ &\quad + \left[\alpha(r,t,T_2)dt - q(r,t,T_2)dZ\right]P(r,t,T_2) + r\left[-P(r,t,T_2) - N \times P(r,t,T_1)\right]dt \end{aligned}$$

We can rewrite N as:

$$N = -\frac{P_r(r,t,T_2)}{P_r(r,t,T_1)} = -\frac{-q(r,t,T_2)P(r,t,T_2)/\sigma(r)}{-q(r,t,T_1)P(r,t,T_1)/\sigma(r)} = -\frac{q(r,t,T_2)P(r,t,T_2)}{q(r,t,T_1)P(r,t,T_1)}$$

We can now express the change in $I(t)$ as:

$$
\begin{aligned}
dI(t) &= N\left[\alpha(r,t,T_1)dt - q(r,t,T_1)dZ\right]P(r,t,T_1) \\
&\qquad +\left[\alpha(r,t,T_2)dt - q(r,t,T_2)dZ\right]P(r,t,T_2) + r\left[-P(r,t,T_2) - N \times P(r,t,T_1)\right]dt \\
&= -\frac{q(r,t,T_2)P(r,t,T_2)}{q(r,t,T_1)P(r,t,T_1)}\left[\alpha(r,t,T_1)dt - q(r,t,T_1)dZ\right]P(r,t,T_1) \\
&\qquad +\left[\alpha(r,t,T_2)dt - q(r,t,T_2)dZ\right]P(r,t,T_2) \\
&\qquad +r\left[-P(r,t,T_2) + \frac{q(r,t,T_2)P(r,t,T_2)}{q(r,t,T_1)P(r,t,T_1)} \times P(r,t,T_1)\right]dt \\
&= -\frac{q(r,t,T_2)}{q(r,t,T_1)}\left[\alpha(r,t,T_1)dt - q(r,t,T_1)dZ\right]P(r,t,T_2) \\
&\qquad +\left[\alpha(r,t,T_2)dt - q(r,t,T_2)dZ\right]P(r,t,T_2) \\
&\qquad +r\left[-P(r,t,T_2) + \frac{q(r,t,T_2)P(r,t,T_2)}{q(r,t,T_1)}\right]dt \\
&= -\frac{q(r,t,T_2)}{q(r,t,T_1)}\alpha(r,t,T_1)dtP(r,t,T_2) + \alpha(r,t,T_2)dtP(r,t,T_2) \\
&\qquad +r\left[-P(r,t,T_2) + \frac{q(r,t,T_2)P(r,t,T_2)}{q(r,t,T_1)}\right]dt \\
&= \left[-\frac{q(r,t,T_2)}{q(r,t,T_1)}\alpha(r,t,T_1) + \alpha(r,t,T_2) + r\left(-1 + \frac{q(r,t,T_2)}{q(r,t,T_1)}\right)\right]P(r,t,T_2)dt
\end{aligned}
$$

The last expression above does not contain a random variable because we have eliminated the dZ terms.

The cost to establish the portfolio is zero, and its return is not random. To preclude arbitrage, the return must therefore be zero:

$$dI(t) = 0$$

$$\left[-\frac{q(r,t,T_2)}{q(r,t,T_1)}\alpha(r,t,T_1) + \alpha(r,t,T_2) + r\left(-1 + \frac{q(r,t,T_2)}{q(r,t,T_1)}\right)\right]P(r,t,T_2)dt = 0$$

$$-\frac{q(r,t,T_2)}{q(r,t,T_1)}\alpha(r,t,T_1) + \alpha(r,t,T_2) + r\left(-1 + \frac{q(r,t,T_2)}{q(r,t,T_1)}\right) = 0$$

$$\frac{\alpha(r,t,T_1)}{q(r,t,T_1)} - \frac{\alpha(r,t,T_2)}{q(r,t,T_2)} + r\left(\frac{1}{q(r,t,T_2)} - \frac{1}{q(r,t,T_1)}\right) = 0$$

$$\frac{\alpha(r,t,T_1)}{q(r,t,T_1)} - \frac{r}{q(r,t,T_1)} = \frac{\alpha(r,t,T_2)}{q(r,t,T_2)} - \frac{r}{q(r,t,T_2)}$$

$$\frac{\alpha(r,t,T_1) - r}{q(r,t,T_1)} = \frac{\alpha(r,t,T_2) - r}{q(r,t,T_2)}$$

We have shown that the two bonds have the same Sharpe ratio. Since we didn't specify T_1 or T_2, we can generalize the result to conclude that all bonds must have the same Sharpe ratio, regardless of maturity. Let's denote the Sharpe ratio with $\phi(r,t)$. The Sharpe ratio can change when the short rate r changes and/or it can change as time t changes, but it is not a function of the time to maturity T. The Key Concept below defines the Sharpe ratio for bonds.

The Greek letter ϕ is called phi.

Sharpe Ratio

The Sharpe ratio for a zero-coupon bond that expires at time T is:

$$\phi(r,t) = \frac{\alpha(r,t,T) - r}{q(r,t,T)}$$

At any point in time, all bonds have the same Sharpe ratio, regardless of maturity.

Keep in mind that this Review Note does not deal with credit risk. The bonds discussed in this Review Note have zero probability of defaulting.

Substituting the expressions for $\alpha(r,t,T)$ and $q(r,t,T)$ into the Sharpe ratio for a zero-coupon bond maturing at time T, we have:

$$\phi(r,t) = \frac{\alpha(r,t,T) - r}{q(r,t,T)}$$

$$\phi(r,t) = \frac{\frac{1}{P}\left[a(r)P_r + \frac{1}{2}[\sigma(r)]^2 P_{rr} + P_t\right] - r}{-\frac{1}{P}\sigma(r)P_r}$$

$$-\phi(r,t)\frac{1}{P}\sigma(r)P_r = \frac{1}{P}\left[a(r)P_r + \frac{1}{2}[\sigma(r)]^2 P_{rr} + P_t\right] - r$$

$$-\phi(r,t)\sigma(r)P_r = a(r)P_r + \frac{1}{2}[\sigma(r)]^2 P_{rr} + P_t - rP$$

$$rP = \frac{1}{2}[\sigma(r)]^2 P_{rr} + \left[a(r) + \sigma(r)\phi(r,t)\right]P_r + P_t$$

This is analogous to the Black-Scholes Equation for stocks that we saw in Chapter 13.

Partial Differential Equation for Bonds of All Maturities

Regardless of the maturity of a bond, the short-term rate times the bond's price is:

$$rP = \frac{1}{2}[\sigma(r)]^2 P_{rr} + \left[a(r) + \sigma(r)\phi(r,t)\right]P_r + P_t$$

Just as we saw in Chapter 13, the equation above implies that the delta-gamma-theta approximation for a change in the price of a bond holds exactly if the interest rate changes by one standard deviation over a very small period of time.

The risk premium of a zero-coupon bond that matures at time T is its expected return minus the risk-free return, which is equal to the Sharpe ratio times the bond's volatility:

$$\text{Risk premium of bond} = \alpha(r,t,T) - r = \phi(r,t)q(r,t,T)$$

Suppose an asset is constructed so that it has a percentage price increase of dr. Such an asset has an expected percentage price increase of $a(r)$ and volatility of $\sigma(r)$:

$$dr = a(r)dt + \sigma(r)dZ$$

Suppose that the dividend rate (or, equivalently, the lease rate) for this asset is $\breve{\delta}(r)$. The expected return on the asset is the expected price increase plus the dividend rate, $a(r) + \breve{\delta}(r)$. Since it has dZ as its source of randomness, this asset must have the same Sharpe ratio as the zero-coupon bonds. But when calculating the Sharpe ratio for this asset, we must multiply the risk premium in the numerator by -1 to account for the fact that the risk premium is negative.

The risk premium is negative because this asset increases in value when the short rate increases. This makes it a natural hedge for a zero-coupon bond, which decreases in value when the short rate increases. Since a zero-coupon bond has a positive risk premium, this asset must have a negative risk premium. Otherwise, it would be possible to combine this asset with a zero-coupon bond to create a risk-free portfolio that earned more than the risk-free rate of return.

The Sharpe ratio and risk premium of an asset that has a percentage price increase of dr are:

Sharpe Ratio: $$\phi(r,t) = \frac{-\left[a(r) + \breve{\delta}(r) - r\right]}{\sigma(r)}$$

Risk Premium: $$a(r) + \breve{\delta}(r) - r = -\sigma(r)\phi(r,t)$$

This risk-premium is worth noting because it can be used to adjust the Itô process for dr to make it into a risk-neutral process.

The textbook refers to this risk premium as "the" risk premium. Although we might prefer to consider it "a" risk premium, we want to be ready in case the exam asks for "the" risk premium. Therefore, the Key Concept below defines this risk premium as "the" risk premium.

Risk Premium

The risk premium is:

$$\text{Risk premium} = -\sigma(r)\phi(r,t)$$

Risk-Neutral Process

We previously stated that the price of zero-coupon bond follows an Itô process:

$$\frac{dP}{P} = \alpha(r,t,T)dt - q(r,t,T)dZ$$

The process can also be described as:

$$\begin{aligned}\frac{dP}{P} &= \alpha(r,t,T)dt - q(r,t,T)dZ \\ &= \alpha(r,t,T)dt + rdt - rdt - q(r,t,T)dZ \\ &= rdt - q(r,t,T)dZ + \left[\alpha(r,t,T) - r\right]dt \\ &= rdt - q(r,t,T)dZ + \frac{q(r,t,T)}{q(r,t,T)}\left[\alpha(r,t,T) - r\right]dt \\ &= rdt - q(r,t,T)\left[dZ - \frac{\alpha(r,t,T) - r}{q(r,t,T)}dt\right]\end{aligned}$$

Let's transform $Z(t)$ into a new Brownian motion:

$$d\tilde{Z}(t) = dZ - \frac{\alpha(r,t,T) - r}{q(r,t,T)}dt$$

$$d\tilde{Z}(t) = dZ - \phi(r,t)dt$$

The process for the bond price can be written as:

$$\frac{dP}{P} = rdt - q(r,t,T)d\tilde{Z}$$

Let's describe the Itô process for *r* using this new Brownian motion:

$$\begin{aligned}dr &= a(r)dt + \sigma(r)dZ \\ &= a(r)dt + \sigma(r)\left[d\tilde{Z} + \phi(r,t)dt\right] \\ &= a(r)dt + \sigma(r)\phi(r,t)dt + \sigma(r)d\tilde{Z} \\ &= \left[a(r) + \sigma(r)\phi(r,t)\right]dt + \sigma(r)d\tilde{Z}\end{aligned}$$

$\tilde{Z}(t)$ is not a martingale under the realistic probability measure, but it is a martingale under the risk-neutral probability measure. If we use the risk-neutral probability measure, then we can assume that the expected return on the bond is *r*. That is, we can assume that the expected change in $\tilde{Z}(t)$ is zero as long as we discount the future value of the bond at the risk-free short rate.

Although Girsanov's Theorem is not specifically cited in Chapter 24, this result is based on Girsanov's Theorem, which tells us that a martingale under one probability measure can be transformed into a martingale under another probability measure.

The drift of the realistic interest rate process is $a(r)$. Subtracting the risk premium from the drift (which is equivalent to adding $\sigma(r)\phi(r,t)$ to the drift) gives us the risk-neutral process for the short-term interest rate.

Risk-Neutral Process for the Interest Rate

The risk-neutral process for the price of a bond is:

$$\frac{dP}{P} = rdt - q(r,t,T)d\tilde{Z}$$

The risk-neutral process for the short rate is:

$$dr = \left[a(r) + \sigma(r)\phi(r,t)\right]dt + \sigma(r)d\tilde{Z}$$

Under the risk-neutral probability measure, $\tilde{Z}(t)$ is a martingale:

$$\tilde{Z}(t) - \tilde{Z}(0) \sim N(0,t)$$

Under the risk-neutral distribution, the Sharpe ratio is always zero since r is substituted for $\alpha(r,t,T)$ in the stochastic differential equation for price in the box above:

Sharpe ratio under the risk-neutral distribution: $\dfrac{\alpha(r,t,T) - r}{q(r,t,T)} = \dfrac{r - r}{q(r,t,T)} = 0$

Under the risk-neutral distribution, the price of a future cash flow is simply the expected discounted value of the cash flow, using the short rates to discount the cash flow. Both the amount of the future cash flow and the future short rates may be unknown at the outset. Suppose that an asset has a value of $V(T)$ at time T and that the asset produces no cash flows prior to time T. The price of the asset at time t is:

$$V(t) = E^*\left[V(T) \times e^{-\int_t^T r(s)ds}\right]$$

For a zero-coupon bond, the value of $V(T)$ is known to be 1. In the Key Concept below, the price of a zero-coupon bond is expressed as an expected value based on the risk-neutral distribution.

Zero-Coupon Bond Price as an Expected Value

The price at time t of a zero-coupon bond expiring at time T is:

$$P(r,t,T) = E^*\left[e^{-R(t,T)}\right]$$

where E^* is the expectation based on the risk-neutral distribution and $R(t,T)$ is the cumulative interest rate:

$$R(t,T) = \int_t^T r(s)ds$$

We've written r as r(s) here to remind us that r is a stochastic process.

We'll use the discrete-time version of this Key Concept when we get to Section 24.4.

*The Key Concept above does **not** say that the price of a zero coupon bond can be found by discounting at the expected interest rate:*

$$P(r,t,T) \neq e^{-E^*[R(t,T)]}$$

Delta-Gamma Approximations for Bonds

We can use Itô's Lemma to find the change in the price of a bond under the risk-neutral distribution:

$$\begin{aligned} dP &= P_r dr + 0.5P_{rr}(dr)^2 + P_t dt \\ &= P_r\left\{\left[a(r)+\sigma(r)\phi(r,t)\right]dt + \sigma(r)d\tilde{Z}\right\} \\ &\qquad +0.5P_{rr}\left\{\left[a(r)+\sigma(r)\phi(r,t)\right]dt + \sigma(r)d\tilde{Z}\right\}^2 + P_t dt \\ &= P_r\left[a(r)+\sigma(r)\phi(r,t)\right]dt + P_r\sigma(r)d\tilde{Z} + 0.5P_{rr}\left[\sigma(r)\right]^2 dt + P_t dt \\ &= \left\{0.5\left[\sigma(r)\right]^2 P_{rr} + \left[a(r)+\sigma(r)\phi(r,t)\right]P_r + P_t\right\}dt + \sigma(r)P_r d\tilde{Z} \end{aligned}$$

The expected change per unit of time is:

$$\frac{E^*[dP]}{dt} = 0.5\left[\sigma(r)\right]^2 P_{rr} + \left[a(r)+\sigma(r)\phi(r,t)\right]P_r + P_t$$

Earlier, we used the fact that all bonds have the same Sharpe ratio to show that:

$$rP = \frac{1}{2}[\sigma(r)]^2 P_{rr} + \left[a(r)+\sigma(r)\phi(r,t)\right]P_r + P_t$$

This implies that:

$$\frac{E^*[dP]}{dt} = rP$$

This result tells us that under the risk-neutral distribution, a bond has an expected return that is equal to the risk-free rate of return.

24.2 Equilibrium Short-Rate Bond Price Models

In this section, the textbook uses dZ and dz interchangeably to represent increments to a standard Brownian motion.

All of the models presented in Section 24.2 satisfy the following partial differential equation from the previous section:

$$rP = \frac{1}{2}[\sigma(r)]^2 P_{rr} + \left[a(r)+\sigma(r)\phi(r,t)\right]P_r + P_t$$

Let's begin by considering (and then rejecting) an overly simplistic model. Suppose that the short-term rate follows arithmetic Brownian motion:

$$dr = a\,dt + \sigma dZ$$

There are several problems with this arithmetic Brownian motion model:

– The short-term rate can become negative.

– The process is not mean-reverting, because the drift, a, is constant.

– The volatility is constant. A more reasonable model would have higher volatility when the short-term rate is high and lower volatility when the short-term rate is low.

The Rendleman-Bartter Model

*The heading of this section contains a misspelling on page 785 of the textbook. The heading should be "The Rend**le**man-Bartter Model," not "The Rend**el**man-Bartter Model."*

In the Rendleman-Bartter model, the short-term rate follows geometric Brownian motion:

$$dr = a r\,dt + \sigma r dZ$$

In the Rendleman-Bartter model, we have:

$$a(r) = a \times r$$
$$\sigma(r) = \sigma \times r$$

The variance of the short-term rate over a small increment of time is:

$$Var\left[r(t+dt) \middle| r(t) \right] = r^2\sigma^2 dt$$

This model solves the first problem and the third problem with the arithmetic model, but it still isn't mean-reverting:

\+ This model does not allow negative interest rates.

– The model is not mean-reverting.

\+ The variance increases with the short-term rate.

The Vasicek Model

The Vasicek model is an Ornstein-Uhlenbeck process, so it is mean-reverting.

Vasicek Model

The Vasicek model of the short-term interest rate is:

$$dr = a(b - r)dt + \sigma dZ$$

The variance of the short-term rate over a small increment of time is:

$$Var\left[r(t + dt) \middle| r(t) \right] = \sigma^2 dt$$

In the Vasicek model, we have:

$$a(r) = a \times (b - r)$$
$$\sigma(r) = \sigma$$

The parameter b is the level to which the short-term rates revert. The parameter a is a positive value that reflects the speed of the reversion to b. The higher a is, the faster the reversion is.

The Vasicek model is mean-reverting, but it exhibits the other two problems of the arithmetic model:

– This model allows negative interest rates.

\+ The model is mean-reverting.

– The variance does not increase with the short-term rate.

In the Vasicek model, the Sharpe ratio is a constant:

$$\phi(r,t) = \phi$$

There are 4 inputs to the Vasicek model: a, b, σ, and ϕ. A yield curve can consist of many more than 4 yields, so the Vasicek model does not have enough inputs to fully describe an existing yield curve. Another way to say this is that the time-zero yield curve cannot be **exogenously prescribed** using the Vasicek model.

Bond Prices in the Vasicek Model

The price of a zero-coupon bond maturing in T years is:

$$P(r,t,T) = A(t,T)e^{-B(t,T)r}$$

The values $A(t,T)$ and $B(t,T)$ depend on whether a is equal to zero.

If $a \neq 0$, then:

$$\bar{r} = b + \phi\frac{\sigma}{a} - 0.5\frac{\sigma^2}{a^2} = \lim_{T\to\infty}\left[\frac{-\ln[P(r,t,T)]}{T-t}\right]$$

$$B(t,T) = \frac{1-e^{-a(T-t)}}{a}$$

$$A(t,T) = e^{\bar{r}\times[B(t,T)-(T-t)] - \frac{[B(t,T)]^2\times\sigma^2}{4a}}$$

If $a = 0$, then:

$$B(t,T) = T - t$$

$$A(t,T) = e^{0.5\sigma\phi(T-t)^2 + \frac{\sigma^2(T-t)^3}{6}}$$

The limit function in the Key Concept above shows that as the maturity of a zero-coupon bond approaches infinity, its yield approaches $\bar{r}$. When $a = 0$, $\bar{r}$ is undefined.

Neither $A(t,T)$ nor $B(t,T)$ depends on the current short rate, $r(t)$. Therefore:

$$P_r = -B(t,T)A(t,T)e^{-B(t,T)r} = -B(t,T)P(r,t,T)$$

In the previous section, we defined:

$$q(r,t,T) = -\frac{1}{P(r,t,T)}P_r\sigma(r)$$

We can now write $q(r,t,T)$ in terms of $B(t,T)$ and σ:

$$q(r,t,T) = -\frac{1}{P(r,t,T)}P_r\sigma(r) = -\frac{1}{P(r,t,T)}\left[-B(t,T)\right]P(r,t,T)\sigma = B(t,T)\sigma$$

This allows us to obtain an expression for $\alpha(r,t,T)$ as well:

$$\phi(r,t) = \frac{\alpha(r,t,T) - r}{q(r,t,T)}$$

$$\phi = \frac{\alpha(r,t,T) - r}{B(t,T)\sigma}$$

$$\alpha(r,t,T) = r + B(t,T)\sigma\phi$$

Vasicek Model Values of α and q

The stochastic differential equation for a bond price in the Vasicek model is parameterized by:

$$\alpha(r,t,T) = r + B(t,T)\sigma\phi$$
$$q(r,t,T) = B(t,T)\sigma$$

The Risk-Neutral Version of the Vasicek Model

This section does not appear in the textbook, but we think it is helpful to understand how to construct the risk-neutral version of the Vasicek model.

The section above described the realistic version of the Vasiceck model. We obtain the same prices whether we use the realistic version or the risk-neutral version. As before, let's assume that the realistic version is parameterized with a, b, σ , and ϕ. We can solve for the equivalent parameters in the risk-neutral version: $\tilde{a}$, $\tilde{b}$, $\tilde{\sigma}$, and $\tilde{\phi}$.

The realistic and the risk-neutral processes for the short rate in the Vasicek model are:

Realistic: $dr = a(r)dt + \sigma(r)dZ = a(b-r)dt + \sigma dZ$

Risk-Neutral: $dr = \left[a(r) + \sigma(r)\phi(r,t)\right]dt + \sigma(r)d\tilde{Z}$

$$= \left[a(b-r) + \sigma\phi\right]dt + \sigma d\tilde{Z}$$

We can rewrite the risk-neutral version so that it takes the following form:

$$dr = \left[a(b-r) + \sigma\phi\right]dt + \sigma d\tilde{Z} = a\left[\left(b + \frac{\sigma\phi}{a}\right) - r\right]dt + \sigma d\tilde{Z}$$

We can rewrite this as:

$$dr = \tilde{a}(\tilde{b} - r)dt + \tilde{\sigma} d\tilde{Z}$$

where:

$$\tilde{a} = a \qquad \tilde{b} = b + \frac{\sigma\phi}{a} \qquad \tilde{\phi} = 0 \qquad \tilde{\sigma} = \sigma$$

We set $\tilde{\phi}$ equal to zero because in the risk-neutral version, the expected return on the bond is equal to the risk-free rate r, and therefore the Sharpe ratio is zero:

$$\tilde{\alpha}(r,t,T) = r \qquad \Rightarrow \qquad \tilde{\phi} = \frac{\tilde{\alpha}(r,t,T) - r}{\tilde{q}(r,t,T)} = 0$$

Using the values of $\tilde{a}$, $\tilde{b}$, $\tilde{\sigma}$, and $\tilde{\phi}$ in place of a, b, σ , and ϕ in the Vasicek bond pricing formula results in the same price as is produced by the realistic version.

Risk-Neutral Version of the Vasicek Model

The risk-neutral parameters of the Vasicek Model are:

$$\tilde{a} = a \qquad \tilde{b} = b + \frac{\sigma\phi}{a} \qquad \tilde{\phi} = 0 \qquad \tilde{\sigma} = \sigma$$

If a, b, σ, and ϕ are replaced by $\tilde{a}$, $\tilde{b}$, $\tilde{\sigma}$, and $\tilde{\phi}$ in the Vasicek model, the bond prices are unchanged. The expected return in the risk-neutral version becomes the risk-free rate of return, but the new standard deviation of the return is the same as in the realistic model:

$$\tilde{\alpha}(r,t,T) = r$$
$$\tilde{q}(r,t,T) = q(r,t,T)$$

The Cox-Ingersoll-Ross Model

The Cox-Ingersoll-Ross Model is commonly abbreviated as CIR.

Cox-Ingersoll-Ross (CIR) Model

The CIR model of the short-term interest rate is:

$$dr = a(b - r)dt + \sigma\sqrt{r}dZ$$

The variance of the short-term rate over a small increment of time is:

$$Var\left[r(t+dt)\middle|r(t)\right] = \sigma^2 r dt$$

The standard deviation of the short-term rate is proportional to the square root of the short-term interest rate:

$$\text{Standard deviation} = \sigma\sqrt{r \times dt}$$

On page 787 of the textbook, just under Equation 24.27, the words "standard deviation" should be substituted for "variance." The same substitution should be made in the third line of text from the bottom of the page.

In the CIR model, we have:

$$a(r) = a \times (b - r)$$
$$\sigma(r) = \sigma\sqrt{r}$$

The CIR model does not exhibit any of the problems exhibited by the arithmetic model:

+ This model does not allow negative interest rates.

+ The model is mean-reverting.

+ The variance increases with the short-term rate.

In the CIR model, the Sharpe ratio depends on an input to the model, $\bar{\phi}$:

$$\phi(r,t) = \bar{\phi}\,\frac{\sqrt{r}}{\sigma}$$

In the last line of page 787 of the textbook, the words "Sharpe ratio" should replace "risk premium."

There are 4 inputs to the CIR model: a, b, σ , and $\bar{\phi}$. As with the Vasicek model, these 4 parameters are not sufficient to fully describe an existing yield curve, so a time-zero yield curve cannot be exogenously prescribed using the CIR model.

The basic formula for the price of a zero-coupon bond in the CIR model is the same as in the Vasicek model, but $A(t,T)$ and $B(t,T)$ are defined differently.

Bond Prices in the CIR Model

The price of a zero-coupon bond maturing in T years is:

$$P(r,t,T) = A(t,T)e^{-B(t,T)r}$$

where:

$$\gamma = \sqrt{(a-\bar{\phi})^2 + 2\sigma^2}$$

$$B(t,T) = \frac{2(e^{\gamma(T-t)} - 1)}{(a-\bar{\phi}+\gamma)(e^{\gamma(T-t)} - 1) + 2\gamma}$$

$$A(t,T) = \left[\frac{2\gamma e^{0.5(a-\bar{\phi}+\gamma)(T-t)}}{(a-\bar{\phi}+\gamma)(e^{\gamma(T-t)} - 1) + 2\gamma}\right]^{\left(\frac{2ab}{\sigma^2}\right)}$$

As the maturity of a zero-coupon bond approaches infinity, its yield approaches:

$$\bar{r} = \frac{2ab}{(a-\bar{\phi}+\gamma)} = \lim_{T\to\infty}\left[\frac{-\ln[P(r,t,T)]}{T-t}\right]$$

As with the Vasicek model, neither $A(t,T)$ nor $B(t,T)$ depends on the current short rate, $r(t)$. Therefore:

$$q(r,t,T) = -\frac{1}{P(r,t,T)}P_r\sigma(r) = -\frac{1}{P(r,t,T)}\left[-B(t,T)\right]P(r,t,T)\sigma\sqrt{r} = B(t,T)\sigma\sqrt{r}$$

In the CIR model, the Sharpe ratio is a function of the short-term rate, so the expression for $\alpha(r,t,T)$ is a bit different from the corresponding expression in the Vasicek model:

$$\phi(r,t) = \frac{\alpha(r,t,T) - r}{q(r,t,T)}$$

$$\bar{\phi}\frac{\sqrt{r}}{\sigma} = \frac{\alpha(r,t,T) - r}{B(t,T)\sigma\sqrt{r}}$$

$$\alpha(r,t,T) = r + B(t,T)\bar{\phi}r$$

Cox-Ingersoll-Ross Model Values of α and q

The stochastic differential equation for a bond price in the CIR model is parameterized by:

$$\alpha(r,t,T) = r + B(t,T)\bar{\phi}r$$

$$q(r,t,T) = B(t,T)\sigma\sqrt{r}$$

The Risk-Neutral Version of the Cox-Ingersoll-Ross Model

This section does not appear in the textbook, but we think it is helpful to understand how to construct the risk-neutral version of a CIR model.

The section above described the realistic version of the CIR model. We obtain the same prices whether we use the realistic version or the risk-neutral version. As before, let's assume that the realistic version is parameterized with a, b, σ, and $\bar{\phi}$. We can solve for the equivalent parameters in the risk-neutral version: $\tilde{a}$, $\tilde{b}$, $\tilde{\sigma}$, and $\tilde{\bar{\phi}}$.

The realistic and the risk-neutral processes for the short rate in the Vasicek model are:

Realistic: $dr = a(r)dt + \sigma(r)dZ = a(b-r)dt + \sigma\sqrt{r}dZ$

Risk-Neutral:
$$\begin{aligned} dr &= \left[a(r) + \sigma(r)\phi(r,t)\right]dt + \sigma(r)d\tilde{Z} \\ &= \left[a(r) + \sigma\sqrt{r}\bar{\phi}\frac{\sqrt{r}}{\sigma}\right]dt + \sigma(r)d\tilde{Z} \\ &= \left[a(b-r) + \bar{\phi}r\right]dt + \sigma\sqrt{r}d\tilde{Z} \end{aligned}$$

We can rewrite the risk-neutral version so that it takes the following form:

$$\begin{aligned} dr &= \left[a(b-r) + \bar{\phi}r\right]dt + \sigma\sqrt{r}d\tilde{Z} = a\left(b - r\frac{a}{a} + \frac{\bar{\phi}r}{a}\right)dt + \sigma\sqrt{r}d\tilde{Z} \\ &= a\left[b - \left(\frac{a-\bar{\phi}}{a}\right)r\right]dt + \sigma\sqrt{r}d\tilde{Z} = a\left(\frac{a-\bar{\phi}}{a}\right)\left[b\left(\frac{a}{a-\bar{\phi}}\right) - r\right]dt + \sigma\sqrt{r}d\tilde{Z} \\ &= \left(a-\bar{\phi}\right)\left[\left(\frac{ab}{a-\bar{\phi}}\right) - r\right]dt + \sigma\sqrt{r}d\tilde{Z} \end{aligned}$$

We can rewrite this as:

$$dr = \tilde{a}(\tilde{b} - r)dt + \tilde{\sigma}\sqrt{r}d\tilde{Z}$$

where:

$$\tilde{a} = a - \bar{\phi} \qquad \tilde{b} = \frac{ab}{a - \bar{\phi}} \qquad \tilde{\bar{\phi}} = 0 \qquad \tilde{\sigma} = \sigma$$

We set $\tilde{\bar{\phi}}$ equal to zero because in the risk-neutral version, the expected return on the bond is equal to the risk-free rate r, and therefore the Sharpe ratio is zero. This implies that the $\tilde{\bar{\phi}}$ parameter is zero:

$$\tilde{\alpha}(r,t,T) = r \quad \Rightarrow \quad \tilde{\phi}(r,t) = \frac{\tilde{\alpha}(r,t,T) - r}{\tilde{q}(r,t,T)} = 0 \quad \Rightarrow \quad \tilde{\bar{\phi}} = \tilde{\phi}(r,t)\tilde{\sigma}\sqrt{r} = 0$$

Using the values of $\tilde{a}$, $\tilde{b}$, $\tilde{\sigma}$, and $\tilde{\bar{\phi}}$ in place of a, b, σ, and $\bar{\phi}$ in the CIR bond pricing formula results in the same price as is produced by the realistic version.

Risk-Neutral Version of the Cox-Ingersoll-Ross Model

The risk-neutral parameters of the CIR Model are:

$$\tilde{a} = a - \bar{\phi} \qquad \tilde{b} = \frac{ab}{a - \bar{\phi}} \qquad \tilde{\bar{\phi}} = 0 \qquad \tilde{\sigma} = \sigma$$

If a, b, σ, and $\bar{\phi}$ are replaced by $\tilde{a}$, $\tilde{b}$, $\tilde{\sigma}$, and $\tilde{\bar{\phi}}$ in the CIR model, the bond prices are unchanged. The expected return in the risk-neutral version becomes the risk-free rate of return, but the new standard deviation of the return is the same as in the realistic model:

$$\tilde{\alpha}(r,t,T) = r$$
$$\tilde{q}(r,t,T) = q(r,t,T)$$

Comparing Vasicek and CIR

With a relatively high level of volatility, the CIR yields tend to be higher than the Vasicek yields. This occurs because the Vasicek yields can be negative.

With a relatively low level of volatility, the mean-reversion effect outweighs the volatility effect, and the Vasicek yields tend to be a bit higher than the CIR yields. Both models produce upward sloping yield curves when volatility is low.

Figure 24.1 on page 789 of the textbook contains a typo. At the bottom of the lower graph, the fourth tick mark should be labeled "20," not "0."

